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Asian Resonance

Fuzzy Mappings In Left K-Sequentially Complete Quasi-Pseudo-Metric Space

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Abstract

Our aim of this paper is to establish common fixed point theorems for fuzzy mappings in left k- sequentially complete quasi-pseudo-metric space and right k- sequentially complete quasi-pseudo-metric space. These are generalization of the corresponding one for metric space given by Heilpern [*2*].

Keywords: Fuzzy Mapping, Fixed Point, Left k - Sequentially Complete Quasi- Pseudo-Metric Spaces.

Introduction

The notion of a fuzzy set was introduced by Zadeh [5] in 1965. Heilpern [2] introduced the concept of a fuzzy mapping, i.e. mapping from an arbitrary set to a certain subfamily of fuzzy sets in a metric linear space. He proved a fixed point theorem for fuzzy contraction mapping. Park and Jeong [3] proved the existence of common fixed points of fuzzy mappings satisfying both contractive type conditions and rational inequalities in complete metric spaces. Gregori and Romaguera [1] extend the theorem of Park and Jeong [3] left k-sequentially complete quasi-pseudo-metric spaces. We extend the theorem of Heilpern.

Preliminaries: -

We shall generally follow the notation of Heilpern.

The set of positive integers is denoted by *N*. A quasi-pseudo-metric on a nonempty set *X* is a non negative real valued function *d* on $X \times X$ such that, for all x, y, z $\in$ *X.*

(i) $d(x, x) = 0$.

(ii) $d(x,z) \le d(x,y) + d(y,z)$.

A pair (X, d) is called a quasi-pseudo-metric space, if *d* is a quasi-pseudo-metric in *X*.

The set $B_{\in}(X)$ = $\{y \in X : d(x, y) < \varepsilon\}$ is the *d*- ball with centre *x* and radius $\varepsilon > 0$. The topology τ (p), having as a base the family of all *d*- balls $B_{\in}(X)$ with $x \in X$ and $\varepsilon > 0$, is the topology on *X* induced by *d.*

If *d* is a quasi-pseudo-metric on *X*, then d^{-1} defined by $d^{-1}(x, y) = d(y, x),$ whenever $x, y \in X,$ is also a quasi-pseudo-metric on *X*. We will denote $B_{\in}^{-1}(X)$ by the d^{-1}-balls with center *x* and radius $\varepsilon > 0$. Only, if confusion is possible we write *d*-closed or d^{-1}-closed. By $d \wedge d^{-1}$ and $d \vee d^{-1}$ we denote min$\{d, d^{-1}\}$ and max$\{d, d^{-1}\}$ respectively.

A sequence $\{x_n\}$ in quasi-pseudo-metric space (X, d) is called left *k*-Cauchy if for each $\varepsilon > 0$ there is $k \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in N$ with $k \le n \le m$.

A sequence $\{x_n\}$ in quasi-pseudo-metric space (X, d) is called right *k*-Cauchy if for each $\varepsilon > 0$ there is $k \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in N$ with $k \le m \le n$.

A quasi-pseudo-metric space (X, d) is said to be left (right) *k*- sequentially complete if each left (right) *k*-Cauchy sequence in (X, d) converges to some point in *X* (with respect to the topology τ (p)).

Let x be a point in X and A, a nonempty subset of X. We define the distance $d(x, A)$ from x to A by

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

Thus $d(x, A) = 0$ if and only if $x \in cl\, A$, the closer of A in X.

Now, let A and B be two nonempty subsets of X. We define the distance $d(A, B)$ from A to B by $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$, and clearly $d(A, B) \neq d(B, A)$ in general.

The *Hausdorff* separation of A from B is define by

$$d_H(A, B) = \sup\{d(a, B) : a \in A\}.$$

Thus we have $d_H(A, B) \geq 0$ with $d_H(A, B) = 0$ if and only if $A \subset cl\, B.$

In addition, the triangle inequality

$$d_H(A, C) \leq d_H(A, B) + d_H(B, C)$$

holds for all non-empty subsets $A, B,$ and C of X. In general, however

$$d_H(A, B) \neq d_H(B, A).$$

The *Hausdroff* distance, deduce from the quasi-pseudo-metric space d, between two nonempty subsets A, B of X is define by

$$H(A, B) = \max\{d_H(A, B), d_H(B, A)\}$$

.

Note that $H(A, B) \geq 0$, $H(A, B) = 0$ if and only if $clA = clB$, clearly

$$H(A, B) = H(B, A)$$, and

$$H(A, C) \leq H(A, B) + H(B, C),$$

for any nonempty subsets A, B and C of X. When d is a metric on X, clearly H is the usual *Hausdorff* distance.

A fuzzy set A on X is a function from X into [0, 1]. If $x \in X$, the function value $A(x)$ is called the grade of membership of x in A.

The α – level set of A denoted by A_α and defined by

$$A_\alpha = \{x : A(x) \geq \alpha\}, \text{ if } \alpha \in [0, 1].$$

$$A_0 = \{\overline{x : A(x) > 0}\}.$$

Where $\overline{B}$ denotes the closure of a set B.

A fuzzy set A is said to be an approximate quantity iff A_α is compact and convex for each $\alpha \in [0, 1]$ and $\sup_{x \in X} A(x) = 1.$

The collection of all fuzzy sets in X is denoted by F (X) and $W(X)$ is the sub-collection of all approximate quantities.

Let (X, d) be a quasi-pseudo-metric space. We define the family of fuzzy sets on $W'(x)$ and $W^*(x)$ as follows.

$W'(X) = \{A \in I^x : A_1$ is non-empty d-closed and d-compact}.

$W^*(X) = \{A \in I^x : A_1$ is non-empty d-closed and d^{-1}-compact}. Let (X, d) be a quasi-pseudo-metric space and let $A, B \in W'(X)$, $\alpha \in [0, 1]$. Then we define

$$P_\alpha(A, B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x, y) = d(A_\alpha, B_\alpha).$$

$$\delta_\alpha(A, B) = \sup_{x \in A_\alpha, y \in B\alpha} d(x, y).$$

$$D_\alpha(A, B) = H(A_\alpha, B_\alpha),$$

where H is the *Hausdorff distance* deduced from the quasi-pseudo-metric space d on X.

$$p(A, B) = \sup_\alpha p_\alpha(A, B).$$

$$\delta(A, B) = \sup_\alpha \delta_\alpha(A, B).$$

$$D(A, B) = \sup_\alpha D_\alpha(A, B).$$

We notice that p_α and δ_α are increasing and non-increasing function of α respectively and

$$p(A, B) = p_1(A, B) = d(A_1, B_1),$$

where $d(A_1, B_1) = \inf\{d(x, y) : x \in A_1, y \in B_1\}$.

Let $A, B \in W(X)$. Then A is said to be more accurate than B, denoted by $A \subset B$, if $A(x) \leq B(x)$ for each $x \in X$. It is easy to verify that the relation"⊂" is partial order determined the family $W(X)$.

Let Y be a quasi- pseudo- metric- space and X be an arbitrary set. F is said to be a Fuzzy mapping if F is a mapping from the set X into $W'(Y)$. i.e. $F(x) \in W'(Y)$ for each $x \in X$. We say that x is a fixed point of the mapping $F : X \to I^x$, if $\{x\} \subset F(x)$.

We used the following lemmas. Which were proved by Gregori.

Lemma 1: - Let $x \in X$ and $A \in W'(X)$. Then $\{x\} \subset A$ if $p_\alpha(x, A) = 0$ for each $\alpha \in [0, 1].$

Lemma 2: - $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ for any $x, y \in X$, $A \in W'(X)$.

Consequently we have

$$p(x, A) \leq d(x, y) + p(y, A).$$

Lemma 3: - if $\{x_0\} \subset A$ then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $A, B \in W'(X)$. Consequently we have $p(x_0, B) = d(x_0, B) \leq D(A, B)$ whenever $\{x_0\} \subset A$.

Lemma 4: - Let (X, d) be a metric linear space $F : X \to W'(X)$ be a fuzzy mapping and $x_0 \in X$. Then there exist $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.

Lemma 5: - Suppose K in a non-empty d^{-1} - Countably compact subset of the Quasi-pseudo-metric space (X, d). If $z \in X$, then there exist $k_0 \in K$ such that

$$d(z, K) = d(z, k_0).$$

We will also use the following lemma Rhoades [4]

Let D denote the closure of the range of d. we shall be concerned with a function Q, defined on d and satisfying the following conditions:

(i) $0 < Q(s) < s$ for each $s \in D/\{0\}$ and $Q(0) = 0$,
(ii) Q is non- decreasing on D, and
(iii) $g(s) = s/(s - Q(s))$ is non increasing on $D/\{0\}$.

Lemma 6: - Q is continuous on D.

Lemma 7: - let $\alpha_1 \in D$. Then, $\lim_{n\to\infty} Q^n(\alpha_1) = 0$.

MAIN THEOREM: -

Theorem: - Let (X, d) be a left k-sequentially complete quasi-pseudo-metric space and let F and G are fuzzy mappings from X to $W^*(X)$ satisfying.

$$D(Fx, Gy) \leq a_1[(d \wedge d^{-1})(x, y) + p(x, F(x))]$$

$$+ a_2[(d \wedge d^{-1})(x, y) + p(y, G(y))]$$

$$+ a_3[p(x, G(y)) + p(y, F(x))]$$

$$+ a_4 \frac{p(x, F(x))p(y, G(y))}{1 + d(x, y)}$$

$$+ a_5 \frac{d(x, y)[p(x, G(y)) + p(y, F(x))]}{d(x, y) + p(y, G(y))}$$

For all $x, y \in X$ and $a_1, a_2, a_3, a_4, a_5 \geq 0$ such that $2a_1 + 2a_2 + 2a_3 + a_4 + a_5 < 1$ and $a_2 + a_3 + a_4 \neq 1$.

Then F and G have a common fixed point.

Proof: - Choose $x_0 \in X$, Let $x_1 \in X$ such that $\{x_1\} \subset F_1(x_0)$ then by lemma 2.2.5 there exists $x_2 \in (G(x_1))_1$ such that $d(x_1, x_2) = d(x_1, (G(x_1))_1)$, since $(G(x_1))_1$ is d^{-1}-countably compact. We then have $d(x_1, x_2) = d(x_1, (G(x_1))_1) < H(x_1, (G(x_1))_1)$ $< D(F(x_0), G(x_1))$

Again we can find $x_3 \in X$ such that $\{x_3\} \subset (F(x_2))_1$ and

$$d(x_2, x_3) \leq D(G(x_1), F(x_2)).$$

Now

$$d(x_1, x_2) \leq D(F(x_0), G(x_1)).$$

$$\leq a_1[(d \wedge d^{-1})(x_0, x_1) + p(x_0, F(x_0))]$$

$$+ a_2[(d \wedge d^{-1})(x_0, x_1) + p(x_1, G(x_1))]$$

$$+ a_3[p(x_0, G(x_1)) + p(x_1, F(x_0))]$$

$$+ a_4 \frac{p(x_0, F(x_0))p(x_1, G(x_1))}{1 + d(x_0, x_1)}$$

$$+ a_5 \frac{d(x_0, x_1)\{p(x_0, G(x_1)) + p(x_1, F(x_0))\}}{d(x_0, x_1) + p(x_1, G(x_1))}$$

$$\leq a_1[(d \wedge d^{-1})(x_0, x_1) + d(x_0, x_1)]$$

$$+ a_2[(d \wedge d^{-1})(x_0, x_1) + d(x_1, x_2)]$$

$$+ a_3[d(x_0, x_2) + d(x_1, x_1)]$$

$$+ a_4 \frac{d(x_0, x_1)d(x_1, x_2)}{1 + d(x_0, x_1)}$$

$$+ a_5 \frac{d(x_0, x_1)\{d(x_0, x_2) + d(x_1, x_1)\}}{d(x_0, x_1) + d(x_1, x_2)}.$$

$$\leq a_1[d(x_0, x_1) + d(x_0, x_1)] + a_2[d(x_0, x_1) + d(x_1, x_2)]$$

$$+ a_3[d(x_0, x_1) + d(x_1, x_2)]$$

$$+ a_4 d(x_1, x_2) + a_5 \frac{d(x_0, x_1)\{d(x_0, x_1) + d(x_1, x_2)\}}{d(x_0, x_1) + d(x_1, x_2)}.$$

$$\leq 2a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 d(x_0, x_1)$$

$$+ a_3 d(x_1, x_2) + a_4 d(x_1, x_2) + a_5 d(x_0, x_1).$$

$\leq (2a_1 + a_2 + a_3 + a_5) d(x_0,\ x_1) + (a_2 + a_3 + a_4)$
$d(x_1,\ x_2)$.

$$d(x_1,\ x_2) \leq \frac{(2a_1 + a_2 + a_3 + a_5)}{(1 - a_2 - a_3 - a_4)} d(x_0,\ x_1).$$

Thus $d(x_1, x_2) \leq h d(x_0, x_1)$,

where $\frac{(2a_1 + a_2 + a_3 + a_5)}{(1 - a_2 - a_3 - a_4)} = h < 1$.

Similarly

$$d(x_2,\ x_3) \leq h d(x_1,\ x_2).$$

Thus

$$d(x_2, x_3) \leq h d(x_1, x_2).$$
$$\leq h^2 d(x_0, x_1).$$

Continuing this process we construct a sequence $\{x_n\}$ such that $\{x_{2n+1}\} \subset F(x_{2n})$ and $\{x_{2n+2}\} \subset G(x_{2n+1})$ for all n = 0, 1 , 2, …
and

$$d(x_n, x_{n+1}) \leq h d(x_{n-1},\ x_n) . \leq h^2 d(x_{n-2},\ x_{n-1}).$$
$$< h^n d(x_0, x_1).$$

Now for $m > n$

$d(x_n,\ x_m) \leq d(x_n,\ x_{n+1}) + (x_{n+1},\ x_{n+2}) +$
$\cdots + d(x_{m-1},\ x_m)$.

$$\leq [h^n + h^{n+1} + h^{n+2} + \cdots + h^{m-1}] d(x_{m-1},\ x_m)$$
$$< \frac{h^n}{1-h} d(x_0, x_1).$$

As $n \to \infty$ $d(x_n,\ x_m) = 0$, [since $h < 1$].

Thus $\{x_n\}$ be a left *k*-Cauchy Sequence. Since *X* is complete therefore $\{x_n\}$ is converges to some *z* in *X.* Now we will prove that *z* is common fixed point of *F* and *G*.

By lemma 2 and.3 we have

$$p_1(z,\ G(z)) \leq p(z,\ G(z)).$$
$$\leq d(z,\ x_{2n+1}) + p(x_{2n+1},\ G(z))$$
$$\leq d(z,\ x_{2n+1}) + p(F(x_{2n}),\ G(z)).$$
$$\leq d(z,\ x_{2n+1}) + D(F(x_{2n}),\ G(z)).$$
$$\leq d(z,\ x_{2n+1}) + a_1 [(d \wedge d^{-1})(x_{2n},\ z) + p(x_{2n},\ F(x_{2n}))$$
$$+ a_2 [(d \wedge d^{-1})(x_{2n},\ z) + p(z, G(z))]$$
$$+ a_3 [p(x_{2n}, G(z)) + p(z, F(x_{2n}))]$$
$$+ a_4 \frac{p(x_{2n}, F(x_{2n})) p(z, G(z))}{1 + d(x_{2n}, z)}$$
$$+ a_5 \frac{d(x_{2n},\ z) \{p(x_{2n},\ G(z)) + p(z,\ F(x_{2n}))\}}{d(x_{2n},\ z) + p(z,\ G(z))}.$$

$$\leq d(z,\ x_{2n+1}) + a_1 [(d \wedge d^{-1})(x_{2n},\ z) + p(x_{2n},\ x_{2n+1})$$
$$+ a_2 [(d \wedge d^{-1})(x_{2n},\ z) + p(z,\ G(z))]$$
$$+ a_3 [p(x_{2n}, G(z)) + p(z,\ x_{2n+1})]$$
$$+ a_4 \frac{p(x_{2n},\ x_{2n+1}) p(z,\ G(z))}{1 + d(x_{2n},\ z)}$$
$$+ a_5 \frac{d(x_{2n},\ z) \{p(x_{2n},\ G(z)) + p(z,\ F(x_{2n}))\}}{d(x_{2n},\ z) + p(z,\ G(z))}.$$

This implies that $p_1(z, G(z)) \to 0$, as $n \to \infty$

Thus by lemma 1 $\{z\} \subset G(z)$.

Similarly it can be shown that $\{z\} \subset F(z)$.

Therefore *z* is a common fixed point of *F* and *G*.

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