Fixed Point Theorem in Probabilistically Convex Manger Space

Kavita Rohit Shrivastava
Deptt of Mathematics and Statisties
Dr. Harisingh Gour Vishwavidyalaya,
Sagar, Madhya Pradesh

Abstract

The main target of this paper have been to apply the concept of probabilistically convexity on menger space and deal a common fixed point theorem by using the concept of compatibility between multi-valued mappings and self mappings in the above context. All the results are the generalization of aforesaid paper of Rhoades [9, 10] Som and Mukherjee [12], Iséki [6], Chang et al. [4] and Lee [8].

Keyword: Probabilistically convex Menger space, weakly commuting mapping, compatible mapping

Introduction

Throughout this paper, we assume that (X, F, Δ) is a Menger space with (ε, λ) - topology τ. Let

CB(X) = {A : A is non-empty closed and bounded subset of X}

C(X) = {A : A is non-empty compact subset of X}.

Definition 1 [3]. Let (X, F, Δ) be a Menger space. $A, B \in CB(X)$ and $x \in X$, we define $F_{x,A}$ and $F_{A,B}$ by

$$F_{x,A}(t) = \sup_{y\in A} F_{x,y}(t) \text{ and } F_{A,B}(t) = \sup_{s<t} \Delta\left(\inf_{x\in A}\sup_{y\in B} F_{x,y}(t),\ \inf_{y\in B}\sup_{x\in A} F_{x,y}(t)\right), \forall t \in R.$$

We say that $F_{x,A}$ is the probabilistic distance from x to A and $F_{A,B}$ is the probabilistic distance from A to B induced by F.

Lemma 2. [4] Let (X, F, Δ) be a Menger space, Δ be a left continuous t-norm, $A \in CB(X)$ and $x, y \in X$. Then we have the following:

1. For any $B \in CB(X)$ and $x \in A$
$\inf_{x\in A}\sup_{y\in B} F_{x,y}(t) \le F_{x,B}(t)$, for all $t \in R$,
2. $F_{x,A}(t) = 1$, for all $t > 0$ if and only if $x \in A$, $F_{x,A}(t_1 + t_2) \ge \Delta\left(F_{x,y}(t_1),\ F_{y,A}(t_2)\right)$, for all $t_1, t_2 > 0$,
3. $F_{x,A}(t)$ is a left-continuous functions at t.

Now, we first consider the properties of an induced Manger space.

Theorem 3 [11].

Let (X, d) be a complete metric space and define $F : X \times X \to D^+$ (set of all distribution function) $F_{x,y}(t) = H(t - d(x, y))$, for $x, y \in E$.

Then the space (X, F, min) with a left continuous t- norm $\Delta = \min$ is a τ- complete Menger space and the topology induced by the metric d coincides with the topology τ. And, for $x \in X$, $K, C \in CB(X)$ we can easily obtain

$$F_{x,K}(t) = H(t - d(x, K)) \text{ and}$$
$$F_{K,C}(t) = H(t - d_H(K, C)).$$

Proposition 4. Let (X, F, Δ) be τ-complete Menger Space induced by the metric d as follows:

$$F_{x,y}(t) = H(t - d(x, y)), \text{ for } x, y \in X,$$

where Δ is a left-continuous t - norm such that $\Delta(a, b) = \min\{a, b\}$.

Let $T : X \to CB(X)$ a multi-valued mapping, then for each $x, y \in X$ and $u_x \in Tx$ there exist a $v_y \in Ty$ such that.

$F_{u_x, v_y}(t) \geq F_{Tx, Ty}(t)$, $t \geq 0$.

Proof. From the compactness of Ty, we can choose $v_y \in Ty$ such that

$d(u_x, v_y) \leq d_H(Tx, Ty)$.

Hence $F_{u_x, v_y}(t) = H(t - d(u_x, v_y))$

$\geq H(t - d_H(Tx, Ty)) = F_{Tx, Ty}(t)$, $t \geq 0$

By Proposition (4) we can easily obtain the following.

Corollary 5. Let (X, F, Δ) be a τ complete Menger space induced by the metric d as follows;

$F_{x, y}(t) = H(t - d(x, y))$, for $x, y \in X$,

where Δ is a left-continuous t-norm such that Δ (a, b) = min {a, b} and

T : X → CB (X) multi-valued mapping. If for each $x, y \in X$

$F_{Tx, Ty}(\phi(t)) \geq F_{x, y}(t)$, $t \geq 0$.

Then for $u_x \in Tx$ there exists $v_y \in Ty$ such that

$F_{u_x, v_y}(\phi(t)) \geq F_{x, y}(t)$, $t \geq 0$,

where $\phi : [0, +\infty) \to [0, +\infty)$ is a function.

Definition 6. A Menger Space (X, F, Δ) is said to be **probabilistically convex** if for any $x, y \in X$ with $x \neq y$, there exists a point $z \in X$, $x \neq z \neq y$ such that

$\Delta(F_{x, z}(t_1), F_{z, y}(t_2)) = F_{x, y}(t_1 + t_2)$.

Lemma 7. Let (X, F, Δ) be a complete probabilistically convex Menger space. Let K be any non-empty closed subset of X. Then for any $x \in K$ and $y \notin K$ there exists a point $z \in \partial K$ (the boundary of K) such that

$\Delta(F_{x, z}(t_1), F_{z, y}(t_2)) = F_{x, y}(t_1 + t_2)$.

Our main theorem is prefaced with the above lemma.

Definition 8. Let K be a non-empty subset of a Menger space (X, F, Δ) and S, T : K → X . The Pair {S, T} is said to be **weakly commuting** if for each $x, y \in K$ such that $x = Sy$ and $Ty \in K$, we have

$F_{Tx, STy}(t) \geq F_{Sy, Ty}(t)$.

Definition 9. Let K be a non- empty subset of a Menger space (X, F, Δ) and S, T : K → X . The pair {S,T} is said to be **compatible** if for every sequence $\{x_n\}$ from K and from relation.

$\lim_{n\to\infty} F_{Tx_n, Sx_n}(t) = 1$

and $Tx_n \in K$, $n \in N$, it follows that

$\lim_{n\to\infty} F_{Ty_n, STx_n}(t) = 1$,

for every sequence $\{y_n\}$ from K such that $y_n = Sx_n$, $n \in N$.

Kaneko and Sessa [7] extended the concept of compatibility for single-valued mapping to a multi-valued mapping as follows :

Definition 10. Let (X, F, Δ) be a Menger space. The mappings

A : X → CB(X) and S : X → X are **compatible** if $SA(x) \in CB(X)$, $\forall x \in X$ and

$\lim_{n\to\infty} F_{SAx_n, ASx_n}(t) = 1$,

whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to M \in CB(X)$ and

$Sx_n \to t \in M$.

In [2] chang defined the family of real functions Φ as follows:

Let $\Phi = \{\phi : R^+ \to R^+, \phi$ is upper semi-continuous with $\phi(x) < x$ for each $x > 0$ and $\phi(0) = 0\}$, where R^+ is the set of all non-negative real numbers.

Lemma 11. [2] Let $\phi \in \Phi$, then there exists a strictly increasing continuous function $\psi : R^+ \to R^+$ such that $\phi(u) \leq \psi(u) < u$ for each $u > 0$, $\lim_{n\to\infty} \psi^{-n}(u) = \infty$ and $\psi(u) > 0$, for each $u > 0$.

Remark 12. [2] In the above case the function ψ is invertible If for each u > 0, we denote $\psi^0(u) = u$ and $\psi^{-n}(u) = \psi(\psi^{-n+1}(u))$, for each $n \in N$, then $\lim_{n\to\infty} \psi^{-n}(u) = \infty$.

Main Results

Byung soo lee [8] proved the following common fixed point theorem for multi-valued mappings on Menger space.

Theorem 13. Let (E, F, Δ) be a τ-complete Menger space with a left continuous t-norm Δ and T : E → C(E) a multi-valued mapping . Suppose that there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the condition (**Φ**) such that for $x, y \in E$, $u_x \in Tx$, there exists a $v_y \in Ty$ satisfying

$F_{u_x, v_y}(\phi(t)) \geq F_{x, y}(t)$, $t \geq 0$.

Then there exists an $x_* \in E$ such that $x_* \in Tx_*$

Now, we are generalizing the above theorem for four non-self mappings in probabilistically convex Menger space.

Theorem 14.

Let (X, F, Δ) be a complete probabilistically convex Menger space and K be a non-empty closed subset of X. If the mappings A, B, S, T : K → X satisfying the inequality

(a) $F_{Ax, By}(\phi t) \geq F_{Sx, Ty}(t)$, $\forall x, y \in X$, $\phi \in \mathbf{\Phi}$, t > 0 and

(b) $\partial K \subseteq SK \cap TK$, $AK \subseteq TK$, $BK \subseteq SK$,

(c) $Sx \in \partial K \Rightarrow Ax \in K$, $Tx \in \partial K \Rightarrow Bx \in K$,

(d) (A, S) and (B, T) are compatible mappings,

(e) A, B, S and T are continuous on K.

Then there exists a unique common fixed point z in K.

Proof. Let $x \in \partial K$. Since $\partial K \subseteq SK$, then there exists a point $x_0 \in K$ such that $x \in Sx_0$. Since $Sx_0 \in \partial K$ and $Sx \in \partial K \Rightarrow Ax \in K$, then $Ax_0 \in K \cap AK \subset TK$. Let $x_1 \in K$ be such that $y_1 = Tx_1 = Ax_0$. Since $x_1 \in K$ and $BK \subset SK$ then there exists a point $x_2 \in K$ such that $Bx_1 = Sx_2 = y_2$. Suppose $y_2 \in K$, otherwise if $y_2 \notin K$, then there exist a point $u \in \partial K$ such that (Lemma 2.1.7)

$$\Delta\left(F_{Tx_1, u}(t_1), F_{u, y_2}(t_2)\right) = F_{Tx_1, y_2}(t_1+t_2).$$

Since $u \in \partial K \subseteq SK$, there exist $x_2 \in K$ such that $u = Sx_2$ and so

$$\Delta\left(F_{Tx_1, Sx_2}(t_1), F_{Sx_2, y_2}(t_2)\right) = F_{Tx_1, y_2}(t_1+t_2).$$

Thus repeating the above argument, we obtain $\{x_n\}$ and $\{y_n\}$ such that

i. $y_{2n} = Ax_{2n-1}$, $y_{2n+1} = Bx_{2n}$,

ii. $y_{2n} \in K \Rightarrow y_{2n} = Sx_{2n}$ or $y_{2n} \notin K \Rightarrow Sx_{2n} \in \partial K$, $\forall$ $t > 0$,

$$\Delta\left(F_{Tx_{2n-1}, Sx_{2n}}(t_1), F_{Sx_{2n}, y_{2n}}(t_2)\right) = F_{Tx_{2n-1}, y_{2n}}(t_1+t_2)$$

iii. $y_{2n+1} \in K$, $y_{2n+1} = Tx_{2n+1}$ or $y_{2n+1} \notin K$, $Sx_{2n+1} \in \partial K$ and $\forall$ $t > 0$,

$$\Delta\left(F_{Sx_{2n}, Tx_{2n+1}}(t_1), F_{Tx_{2n+1}, y_{2n+!}}(t_2)\right) = F_{Sx_{2n}, y_{2n+1}}(t_1+t_2).$$

We denote

$P_0 = \{Sx_{2i} \in \{Sx_{2n}\} : Sx_{2i} = y_{2i}\}$,

$P_1 = \{Sx_{2i} \in \{Sx_{2n}\} : Sx_{2i} \neq y_{2i}\}$,

$Q_0 = \{Tx_{2i+1} \in \{Tx_{2n+1}\} : Tx_{2i+1} = y_{2i+1}\}$,

$Q_1 = \{Tx_{2i+1} \in \{Tx_{2n+1}\} : Tx_{2i+1} \neq y_{2i+1}\}$.

First, we show that $(Sx_{2n}, Tx_{2n+1}) \notin P_1 \times Q_1$ and $(Tx_{2n-1}, Sx_{2n}) \notin Q_1 \times P_1$

If $Sx_{2n} \in P_1$, then $y_{2n} \neq Sx_{2n}$ and we have $Sx_{2n} \in \partial K$ which implies that

$y_{2n+1} = Ax_{2n} \in K$. Hence $y_{2n+1} = Tx_{2n+1} \in Q_0$. Similarly, one can argue that

$(Tx_{2n-1}, Sx_{2n}) \notin Q_1 \times P_1$.

There are three possibilities.

Case 1. If $(Sx_{2n}, Tx_{2n+1}) \in P_0 \times Q_0$, then

$$F_{Sx_{2n}, Tx_{2n+1}}(\phi t) = F_{y_{2n}, y_{2n+1}}(\phi t)$$

$$= F_{Ax_{2n-1}, Bx_{2n}}(\phi t) \geq F_{Sx_{2n-1}, Tx_{2n}}(t).$$

Case 2. If $(Sx_{2n}, Tx_{2n+1}) \in P_0 \times Q_1$ then

$$\Delta\left(F_{Sx_{2n}, Tx_{2n+1}}(\phi t), F_{Tx_{2n+1}, y_{2n+1}}(\phi t)\right) = F_{Sx_{2n}, y_{2n+1}}(2\phi t)$$

which turn yields

$$F_{Sx_{2n}, Tx_{2n+1}}(\phi t) \geq F_{Sx_{2n}, y_{2n+1}}(2\phi t)$$

$$= F_{y_{2n}, y_{2n+1}}(2\phi t) = F_{Ax_{2n-1}, Bx_{2n}}(2\phi t)$$

$$\geq F_{Sx_{2n-1}, Tx_{2n}}(t).$$

Case 3. If $(Sx_{2n}, Tx_{2n+1}) \in P_1 \times Q_0$ then $Tx_{2n-1} = y_{2n-1}$. Hence, proceeding as in Case 1 we have,

$$F_{Sx_{2n}, Tx_{2n+1}}(2\phi t) = F_{Sx_{2n}, y_{2n+1}}(2\phi t)$$

$$\geq \Delta\left(F_{Sx_{2n}, y_{2n}}(\phi t), F_{y_{2n}, y_{2n+1}}(\phi t)\right)$$

$$\geq \Delta\left(F_{Sx_{2n}, y_{2n}}(\phi t), F_{Sx_{n-1}, Tx_{2n}}(t).\right)$$

Since $\Delta\left(F_{Tx_{2n-1}, Sx_{2n}}(t), F_{Sx_{2n}, y_{2n}}(t)\right) = F_{Tx_{2n-1}, y_{2n}}(2t)$ Then $F_{Sx_{2n}, Tx_{2n+1}}(2\phi t)$

$$\geq \Delta\left(F_{Tx_{2n-1}, y_{2n}}(2\phi t), F_{Sx_{2n-1}, Tx_{2n}}(t)\right)$$

$$= \Delta\left(F_{y_{2n-1}, y_{2n}}(2\phi t), F_{Sx_{2n-1}, Tx_{2n}}(t)\right)$$

$$\geq \Delta\left(F_{Sx_{2n-1}, Tx_{2n}}(t), F_{Sx_{2n-1}, Tx_{2n}}(t)\right)$$

$$\geq \Delta F_{Sx_{2n-1}, Tx_{2n}}(t).$$

Thus in all the cases we put $z_{2n} = Sx_{2n}$, $z_{2n+1} = Tx_{2n+1}$, we have

$$F_{z_{2n}, z_{2n+1}}(\phi t) \geq F_{z_{2n-1}, z_{2n}}(t)$$

$$F_{z_{2n}, z_{2n+1}}(t) \geq F_{z_{2n-1}, z_{2n}}(\phi^{-1} t), \ \forall\, t > 0.$$

Therefore further, we will obtain

$$F_{z_{2n}, z_{2n+1}}(t) \geq F_{z_{2n-1}, z_{2n}}(\phi^{-1} t) \geq F_{z_{2n-2}, z_{2n-1}}(\phi^{-2} t) \geq \ldots\ldots \geq F_{z_0, z_1}(\phi^{-n} t).$$

Taking limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} F_{z_{2n}, z_{2n+1}}(t) = 1.$$

Now,

$$F_{z_{2n}, z_{2n+p}}(t) = \left(F_{z_{2n}, z_{2n+1}}(t/p), F_{z_{2n+1}, z_{2n+2}}(t/p), \ldots\ldots, F_{z_{2n+p-1}, z_{2n+p}}(t/p)\right)$$

taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} F_{z_{2n}, z_{2n+p}}(t) = 1.$$

This implies that $\{z_n\}$ is a cauchy sequence and hence converges to a point z, consequently subsequence

$\{z_{2n}\} = \{Sx_{2n}\} \to z$,

$\{z_{2n+1}\} = \{Tx_{2n+1}\} \to z$.

Since (B, T) is compatible

$$\lim_{n\to\infty} F_{Bx_{2n-1}, Tx_{2n-1}}(t) = 1$$

$$\Rightarrow \lim_{n\to\infty} F_{TSx_{2n}, BTx_{2n-1}}(t) = 1,$$

by the continuity of B and T then

$$F_{Tz, Bz}(t) = 1$$

i.e. $Tz = Bz$.

Similarly, the continuity of S and A and compatibility of (A, S) lead to

$Sz = Az$.

Again, $F_{Sz, Tx_{2n+1}}(\phi t) = F_{Az, Bx_{2n+1}}(\phi t) \geq F_{Sz, Tx_{2n+1}}(t)$.

Taking $n \to \infty$

$F_{Sz, z}(\phi t) \geq F_{Sz, z}(t)$

which implies Sz = z = Az.

Again,

$F_{Sx_{2n}, Tz}(\phi t) = F_{Ax_{2n}, Bz}(\phi t) \geq F_{Sx_{2n}, Tz}(t)$.

Taking $n \to \infty$

$F_{z, Tz}(\phi t) \geq F_{z, Tz}(t)$

which implies that z = Tz = Bz.

Hence z is a common fixed point of A, B, S and T in X.

Uniqueness

Let w be another fixed point of A, B, S, T i.e. w ≠ z and

w = Aw = Bw = Sw = Tw, then

$F_{z, w}(\phi t) = F_{Az, Bw}(\phi t) \geq F_{Sz, Tw}(t) = F_{z, w}(t)$.

This implies z = w.

Thus z is the unique common fixed point of A, B, S and T.

This completes the proof.

Remark 15. If we will do Theorem 14 in metrically convex complete metric space and A = B and S = T then we will obtain the result of Ahmad and Asad [1].

If we take S = T = i_k (i_k ; identity mapping in K) and $\phi(t) = ht$, $h \in (0, 1)$ in Theorem 14 we get following corollary.

Corollary 16. Let (X, F, Δ) be a complete probabilistically convex menger space and K a non empty closed subset of X. Let A, B : K→ X satisfying inequality

$$F_{Ax, By}(ht) \geq F_{x, y}(t) \ \forall \ x, y \in X, \phi \in \Phi$$

and t > 0.

Then A and B have a unique common fixed point.

We can proof this theorem trivially.

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