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Pure-Supplimented Modules and Rings

D.S. Singh
Deptt. of Pure & Applied Mathematics
Guru Ghasidas Vishwavidyalaya,
Bilaspur, Chattisgarh

Abstract

Let R be an associative ring with identity and M be unital non zero right R-module. M is called H-supplemented module if given any submodule A of M there exist a direct summand submodule D of M such that M = A + X iff M = D + X where X is a submodule of M. In this paper we will give a generalization for H-supplemented which is called pure-supplemented module. An R-module M is called pure-supplemented module if given any submodule A of M there exists a pure submodule P of M such that M = A + X iff M = P + X. Equivalently, for every submodule A of M there exist a pure submodule P of M such that $\frac{A+P}{P} << \frac{M}{P}$ and $\frac{A+P}{A} << \frac{M}{A}$.

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Introduction

Let R be an associative ring with identity and M be a non zero unital right R-module. A submodule N of M is called a small submodule of M, denoted by $N << M$, if $N + L \neq M$ for any proper submodule L of M [1]. Let U be a submodule of M, a submodule V of M is called supplement of U if V is minimal element in the set of submodules $L < M$ with $U + L = M$ equivalently $U + V = M$ and $U \cap V << V$. An R-module M is called supplemented if every submodule of M has supplement in M [2].

M is called H-supplemented module if given any submodule A of M there exist a direct summand D of M such that $M = A + X$ iff $M = D + X$ [3]. M is called weakly supplemented module if for each submodule A of M there exists a submodule D of M such that $M = A + D$ and $A \cap D << M$ [2].

An R-module M is called lifting if for every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 << M$ [2]. M is called pure-supplemented module if given any submodule A of M there exists a pure submodule P of M such that $M = A + X$ iff $M = P + X$. Equivalently, for every submodule A of M there exist a pure submodule P of M such that $\frac{A+P}{P} << \frac{M}{P}$ and $\frac{A+P}{A} << \frac{M}{A}$.

Since every direct summand submodule is pure then it is clear that each H-supplemented module is pure-supplemented. In this work, we review the concept of pure-supplemented module and we discuss some of the basic properties of these types of modules.

1. Pure-supplemented modules:

In this section we introduce the pure-supplemented module as a generalization of H-supplemented.

Definition1.1: [2]. Let M be an R-module. M is called H-supplemented module if given any submodule A of M there exists a direct summand submodule D of M such that $M = A + X$ iff $M = D + X$.

Definition1.2: [3]. Let M be a R-module. P is called pure submodule of M if $KM \cap KP$ for every ideal K in R.

Remark1.3: [3]. (i) Any direct summand submodule is pure submodule in M.

(ii) If $H \leq M$ and $K \leq H$ such that H is pure in M and K is pure in H then K is pure in M.

(iii) If A is pure submodule of M and K is pure submodule of N then $A \oplus K$ is pure of $M \oplus N$.

Definition1.4: Let M be a module. M is called pure-supplemented module if given any submodule A of M there exists a pure submodule P of M such that $M = A + X$ iff $M = P + X$.

Since every direct summand submodule is pure then it is clear that each H-supplemented module is pure-supplemented.

Remark1.5:

(i) Every hollow module is pure-supplemented module.

(ii) Every lifting module is pure-supplemented module.

(iii) Every P-supplemented is weakly supplemented.

Proof: (i) Since every hollow module is H-supplemented module then it is pure-supplemented module.

(ii) Let A be submodule of M there exists $K \leq A$, $M = N \oplus K$ where $N \leq M$ and $N \cap A << M$ then M = A + N iff M = K+N where K is pure since K is direct summand submodule.

(iii) Let N be submodule of M such that N+X = M, to show that $N \cap X << M$. Let $(N \cap X) + L = M$, since M pure-supplemented there exists a pure submodule P of M such that $N \cap X + L = M$ iff M = P+L iff $N \cap X + P \leq L$ then $N \cap X \leq L$ hence L = M and $N \cap X << M$.

Definition1.6: Let M be a module. M is called pure-lifting module if for every submodule A of M there exists a pure submodule P of M, $P \leq A$ such that $M = P + X$ with $A \cap X << X$.

It is clear that every lifting and semisimple module is pure-lifting module.

Theorem 1.7: The following are equivalent for an R-module M:

(i) M is pure-lifting module.

(ii) Every submodule N of M can be written as N = A+S where A is pure in M and S<<M.

(iii) For every submodule N of M there exists a pure submodule A of N such that M = A+K and $\frac{N}{A} << \frac{M}{A}$.

Proof: $(i) \Rightarrow (ii)$ Let N be a submodule of M, then M = P+X, $P \leq N$ pure in M and $N \cap X << X$ hence $N \cap X << M$. Now

$$N = N \cap M = N \cap (P + X)$$

$$= N \cap P + N \cap X$$

$$= P + (N \cap X)$$

Take A = P, therefore $S = N \cap X$.

$(ii) \Rightarrow (iii)$ Let N be a submodule of M. By (ii) N = A+S where A is pure in M and $S << M$. suppose $\frac{M}{A} = \frac{N}{A} + \frac{L}{A}$ then $\frac{M}{A} = \frac{A+S}{A} + \frac{L}{A}$, thus A+S+L = M, by (ii) since S<<M then A+L = M.

$(iii) \Rightarrow (i)$ Let N be a submodule of M, there exists a pure submodule A of N such that M = A+K and $\frac{N}{A} << \frac{M}{A}$ to prove that $N \cap K << K$. Suppose that $N \cap K + B = K$ where $B \leq K$ then $M = A + K = A + N \cap K + B$ thus

$$\frac{M}{A} = \frac{A + (N \cap K) + B}{A}$$

$$= \frac{N \cap K + A}{A} + \frac{A + B}{A}$$

$$= \frac{N}{A} + \frac{A + B}{A}$$

, since $\frac{N}{A} << \frac{M}{A}$ then $\frac{A+B}{A} = \frac{M}{A}$ thus A+B = M then and hence B = K thus $N \cap K << K$.

Proposition1.8: Every pure-lifting module is pure-supplemented module.

Proof: Let M be pure-lifting module and A be a submodule of M. Suppose that M = A+Y then M = K+L where $K \leq A$ and K pure in M and $A \cap L << M$. Now

$A = A \cap M = A \cap (K + L) = K + A \cap L$ then $M = A + X = K + A \cap L + X$ since $A \cap L << M$ then M = K+X thus M = A+X. Since $K \leq L$ therefore M is pure-supplemented module.

Proposition1.9: Let M be an R-module then M is pure-supplemented module iff for every submodule A of M there exist a pure submodule P of M such that $\frac{A+P}{P} << \frac{M}{P}$ and $\frac{A+P}{A} << \frac{M}{A}$.

Proof: Let M be a pure-supplemented module and $A \leq M$ then there exist a pure submodule P such that M = A+X iff M = P+X.

Suppose that $\frac{A+P}{P} + \frac{L}{P} = \frac{M}{P}$ then $\frac{A+P+L}{P} \frac{M}{P}$ thus $\frac{A+L}{P} = \frac{M}{P}$. Then A+L = M.

M is pure-supplemented, A+L = M = P+X, $P \leq L$ then $A + X \leq L$. Now $M \leq L$ and L = M thus $\frac{L}{P} = \frac{M}{P}$ therefore $\frac{A+P}{P} << \frac{M}{P}$ similarly $\frac{A+P}{A} << \frac{M}{A}$.

Converse: Let A be a submodule of M then there exists a pure submodule P of M such that

$\frac{A+P}{P} << \frac{M}{P}$ and $\frac{A+P}{A} << \frac{M}{A}$. If M = A+X then $\frac{A+X}{P} = \frac{M}{P} \Rightarrow \frac{M}{P} = \frac{A+P}{P} + \frac{X+P}{P}$

But $\frac{A+P}{P} << \frac{M}{P}$ then $\frac{M}{P} = \frac{X+P}{P}$ thus M = X+P. In the same way one can show that if M = X+P then M =A+X.

Proposition1.10: Let M be pure-supplemented module and A be a submodule of M. If for every pure submodule P of M, $\frac{A+P}{A}$ is pure in $\frac{M}{A}$ then $\frac{M}{A}$ is pure-supplemented module.

Proof: Let $\frac{N}{A} \leq \frac{M}{A}$ and $\frac{M}{A} = \frac{N}{A} + \frac{X}{A}$ where $A \leq X$ then M = N+X iff M =P+X where P is pure in M. Therefore $\frac{M}{A} = \frac{P+X}{A} = \frac{P+A}{A} + \frac{X}{A}$ by assumption $\frac{A+P}{A}$ is pure in $\frac{M}{A}$ hence $\frac{M}{A}$ is pure-supplemented.

Recall that a submodule A of M is called fully invariant if for every $f \in End_R(M), f(X) \subseteq X$. A module M is called dou module if every submodule id fully invariant and M is called distributive iff for every submodules K, L, N of M we have $N+(K \cap L) = (N+K) \cap (N+L)$ or $N \cap (K+L) = (N \cap K) + (N \cap L)$ [2].

Corollary 1.11: Let M be a distributive pure-supplemented module then $\frac{M}{A}$ is pure-supplemented module for every submodule A of M.

Proof: Let D be direct summand of M, then $M = D \oplus K$ for some K submodule of M. Now $\frac{M}{A} = \frac{D+A}{A} + \frac{K+A}{A}$ and $A = A + (D \cap K) = (A+D) \cap (A+K)$ (M is distributive) then $\frac{M}{A} = \frac{D+A}{A} \oplus \frac{K+A}{A}$ hence $\frac{D+A}{A}$ is direct summand of $\frac{M}{A}$ is pure in $\frac{M}{A}$ thus by proposition (1.10) we get $\frac{M}{A}$ is pure-supplemented.

Corollary 1.12: Let A be a submodule of M and $eA \subseteq A$ for all $e^2 = e \in End_R(M)$ then $\frac{M}{A}$ is pure supplemented. In particular for every fully invariant submodule Y of M, $\frac{M}{Y}$ is pure-supplemented.

Proof: Let D be a direct summand of M. Consider the projection map $e : M \to D$ then $e^2 = e \in End_R(M)$, $eA \subseteq A$ and hence $eA = A \cap D$. Since D is direct summand of M then $M = D \oplus K$, $K \leq M$ hence $A = (A \cap D) \oplus (A \cap K)$.

Now $\frac{D+A}{A} = \frac{D \oplus (A \cap K)}{A}$ and $\frac{K+A}{A} = \frac{K \oplus (A \cap D)}{A}$ hence $M = D \oplus K = D + A + K + A = (D \oplus (A \cap K)) + K + A.$ Then $\frac{M}{A} = \frac{D \oplus (A \cap K)}{A} + \frac{K+A}{A}$. Since $(D \oplus (A \cap K)) \cap K + A = (A \oplus K) \cap (A \cap D) = A$. Then $\frac{M}{A} = \frac{D \oplus (A \cap K)}{A} \oplus \frac{K+A}{A}$. Hence $\frac{K+A}{A}$ is direct summand of $\frac{M}{A}$ then and by (prop. 1.10) $\frac{M}{A}$ is pure-supplemented.

Completely pure-supplemented modules

We call a module M is completely pure-supplemented module. If every direct summand of M is pure-supplemented.

Proposition2.1: Every lifting is completely pure-supplemented module.

Proof: Let N is a direct summand of M, and $L \leq N$. Since M is lifting therefore $L = K \oplus A$ where K is a direct summand of M and A<<M [2]. Let $X \leq M$ with N = L+X = K+A+X, since A direct summand of L and A<<M then by (1.3, ii) A<<L. Then N= K+X hence N = L+X iff N = K+X. An R-module M has PSP if the sum of any two pure submodule of M is pure [4].

Proposition2.2: Let M be pure-supplemented module and M has PSP then M is completely pure-supplemented module.

Proof: Let N be a direct summand of M. We show that N is pure-supplemented. $M = N \oplus K, K \leq M$. Assume P is pure in M then by assumption N+P pure in M, Now $M = N + P \oplus K$ then $\frac{M}{A} = \frac{K+P}{K} + \frac{K+N}{K}$ then $\frac{M}{K}$ is pure-supplemented (prop.1.10), but $\frac{M}{K} \cong N$ then N pure-supplemented module.

Proposition2.3: If an R-module M has PSP and $M = M_1 \oplus M_2$ is dou module, then M is pure-supplemented iff M_1 and M_2 are pure-supplemented modules.

Proof: Since M_1 and M_2 are fully invariant submodule hence M_1 and M_2 are pure-supplemented modules (coro. 1.12)

Converse: Assume M_1 and M_2 are pure-supplemented modules and let $L \leq M$ then there exist a pure submodule P_1 of M such that $M_1 = P_1 + X$ iff $M_1 = P_1 + X$ for any submodule X of M_1. And there exist a pure submodule P_2 of M such that $M_2 = (L \cap M_2) + Y$ iff $M_2 = P_2 + Y$ for any submodule Y of M_2.

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