

# Some Common Fixed Point Results in Fuzzy, Fuzzy 2 and Fuzzy 3-Metric Spaces

## Abstract

Some common fixed point results are proved in fuzzy metric, fuzzy 2-metric, fuzzy 3-metric spaces for weak compatible mappings and rational expression.

**Keywords:** Fuzzy Metric Space, Fuzzy 2-Metric Space, Fuzzy 3-Metric Spaces, Weak Compatible Mappings.

## Mathematics Subject Classification

47H10, 54H25.

## Introduction

More than 52 years back the concept of "Fuzzy Sets" was introduced in 1965, by Zadeh [34]. In continuation Zadeh [35], developed the theory of. "Probability measures of fuzzy events", and produced some important results. Not only results were important it opens a vast windows in the field of fixed point theory with a new approach. After that many authors have expansively developed the theory of fuzzy sets and its applications, specially, Deng [8], Kaleva and Seikkala [20], Kramosil and Michalek [21], have introduced the concept of fuzzy metric spaces in different ways. Many authors [1], [6], [9], [15], [16], [18], [19], [22], [24], [27], [28] have also studied the fixed point theory in the fuzzy metric spaces and [2], [3], [4], [5], [17], [23], [31] have studied for fuzzy mappings which opened a new window for further development of analysis in such spaces and such mappings. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors. Gahler in a series of papers [11], [12], [13] investigated 2-metric spaces. Sharma, Sharma and Iseki [26] studied for the first time contraction type mappings in 2-metric space. Recently Yadava, Rajpoot and Bhardwaj [32], [33] have also worked on 2-Metric spaces and 2- Banach spaces for rational expressions. Very recently in 2016 Garg, S. and Shukla, M. K. proved some fixed point result in complete fuzzy metric space, fuzzy 2-metric space and fuzzy 3-metric space [29], [30] for weak compatible mappings

## Aim of the Study

Our aim in this present paper is to generalize the earlier results of fisher [10] and all others also we extend this result to fuzzy 2 and 3-metric spaces by obtaining some common fixed point theorems on fuzzy metric spaces. Till now work on this approach is not reported.

## Preliminaries

We know that 2-metric space is a real valued function of a point triples on a set  $X$ , which abstract properties were suggested by the area function in Euclidean spaces. Before going to our main results the following basic definitions, lemmas and already established results are required:

### Definition 2.1 [25]

A binary operation  $*$ :  $[0,1] \times [0,1] \rightarrow [0,1]$  is called a continuous  $t$ -norm if  $([0,1], *)$  is an abelian Topological monodies with unit 1 such that  $a * b \geq c * d$  whenever  $a \geq c$  and  $b \geq d$  for all  $a, b, c, d \in [0, 1]$

Example of  $t$ -norm are  $a * b = a b$  and  $a * b = \min \{a, b\}$

### Definition 2.2 [21]

The 3 - tuple  $(X, M, *)$  is called a fuzzy metric space if  $X$  is an arbitrary set,  $*$  is a continuous  $t$ -norm and  $M$  is a fuzzy set in  $X^2 \times [0, \infty)$  satisfying the following conditions for all  $x, y, z \in X$  and  $s, t > 0$ ,

$$(FM-1): M(x, y, 0) = 0$$

$$(FM-2): M(x, y, t) = 1, \forall t > 0 \Leftrightarrow x = y$$

$$(FM-3): M(x, y, t) = M(y, x, t)$$

$$(FM-4): M(x, z, t + s) \geq M(x, y, t) * M(z, y, s)$$



**Rashmi Pathak**

Research Scholar,  
Deptt.of Mathematics and  
Computer Science,  
U.T.D, Rani Durgavati University,  
Jabalpur, M. P.

**Manoj Kumar Shukla**

Professor,  
Deptt.of Mathematics,  
Institute for Excellence in Higher  
Education,  
Bhopal, (M.P.)

(FM-5):  $M(x, y, a): [0,1] \rightarrow [0,1]$  is left continuous  
In what follows  $(X, M, *)$  will denote a fuzzy metric space.

Note that  $M(x, y, t)$  can be thought of as the degree of nearness between  $x$  and  $y$  with respect to  $t$ . We identify  $x = y$  with  $M(x, y, t) = 1$  for all  $t > 0$  and  $M(x, y, t) = 0$  with  $\infty$ .

**Example 2.1** [14]

Let  $(X, d)$  be a metric space.

Define

$$a * b = a b, \text{ or } a * b =$$

$\min\{a, b\}$  and for all  $x, y \in X$  and  $t > 0$ ,

$$M(x, y, t) = \frac{t}{t+d(x,y)} \dots \dots \dots (2.2.1.1)$$

Then  $(X, M, *)$  is a fuzzy metric space. We call this fuzzy metric  $M$  induced by the metric  $d$  the standard fuzzy metric.

**Definition 2.3** [15]

Let  $(X, M, *)$  is a fuzzy metric space.

(i) A sequence  $\{x_n\}$  in  $X$  is said to be convergent to a point  $x \in X$ ,

$$\lim_{n \rightarrow \infty} M(x, y, t) = 1$$

(ii) A sequence  $\{x_n\}$  in  $X$  is called a Cauchy sequence if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1 \forall t > 0 \text{ And } p > 0$$

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is called complete.

Let  $(X, M, *)$  is a fuzzy metric space with the following condition.

$$(FM-6): \lim_{n \rightarrow \infty} M(x, y, t) = 1 \forall x, y \in X$$

**Definition 2.4**

A function  $M$  is continuous in fuzzy metric space iff whenever

$$x_n \rightarrow x, y_n \rightarrow y \Rightarrow \lim_{n \rightarrow \infty} M(x_n, y_n, t) \rightarrow M(x, y, t)$$

**Definition 2.5**

Two mappings  $A$  and  $S$  on fuzzy metric space  $X$  are weakly commuting if and only if  $M(ASu, SAu, t) \geq M(Au, Su, t) \forall u \in X$

**Definition 2.6**

A binary operation  $*$ :  $[0,1] \times [0,1] \times [0,1] \rightarrow [0,1]$  is called a continuous  $t$ -norm if  $([0,1], *)$  is an abelian topological monodies with unit 1 such that

$$a_1 * b_1 * c_1 \geq a_2 * b_2 * c_2 \text{ whenever}$$

$$a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$$

$c_2$  for all  $a_1, a_2, b_1, b_2$  and  $c_1, c_2$  are in  $[0,1]$ .

**Definition 2.7**

The 3-tuple  $(X, M, *)$  is called a fuzzy 2-metric space if  $X$  is an arbitrary set,  $*$  is continuous  $t$ -norm and  $M$  is fuzzy set in  $X^3 \times [0, \infty)$  satisfying the following conditions:

$$(FM-1): M(x, y, z, 0) = 0$$

$$(FM-2): M(x, y, z, t) = 1, \forall t > 0 \Leftrightarrow x = y$$

$$(FM-3): M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t) \text{ symmetry about three variable}$$

$$(FM-4): M(x, y, z, t_1, t_2, t_3) \geq M(x, y, u, t_1) * M(z, u, z, t_2) * M(u, y, z, t_3)$$

$$(FM-5): M(x, y, z): [0,1] \rightarrow [0,1] \text{ is left continuous, } x, y, z, u \in X, t_1, t_2, t_3 > 0$$

**Definition 2.8**

Let  $(X, M, *)$  be a fuzzy 2-metric space.

(1) A sequence  $\{x_n\}$  in fuzzy 2-metric space  $X$  is said to be convergent to a point  $x \in X$ ,

$$\lim_{n \rightarrow \infty} M(x_n, x, a, t) = 1 \text{ for all } a \in X \text{ and } t > 0$$

(2) A sequence  $\{x_n\}$  in fuzzy 2-metric space  $X$  is called a Cauchy sequence, if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, a, t) = 1 \text{ for all } a \in X \text{ and } t, p > 0$$

(3) A fuzzy 2-metric space in which every Cauchy sequence is convergent is called complete.

**Definition 2.9**

A function  $M$  is continuous in fuzzy 2-metric space, iff whenever

$$x_n \rightarrow x, y_n \rightarrow y \text{ Then } \lim_{n \rightarrow \infty} M(x_n, y_n, a, t) \rightarrow M(x, y, a, t) \text{ for all } a \in X, t > 0$$

**Definition 2.10**

Two mappings  $A$  and  $S$  on fuzzy 2-metric space  $X$  are weakly commuting iff  $M(ASu, SAu, a, t) \geq M(Au, Su, a, t)$

**Definition 2.11**

A binary operation  $*$ :  $[0, 1]^4 \rightarrow [0, 1]$  is called a continuous  $t$ -norm if  $([0, 1], *)$  is an abelian topological monoid with unit 1 such that  $\forall a_1, a_2, b_1, b_2, c_1, c_2$  and  $d_1, d_2 \in [0, 1]$   $a_1 * b_1 * c_1 * d_1 \geq a_2 * b_2 * c_2 * d_2$  where  $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$  and  $d_1 \geq d_2$ .

**Definition 2.12**

The 3-tuple  $(X, M, *)$  is called a fuzzy 3-metric space if  $X$  is an arbitrary set,  $*$  is a continuous  $t$ -norm monoid and  $M$  is a fuzzy set in  $X^4 \times [0, \infty]$  satisfying the following conditions:

$$(FM' - 1): M(x, y, z, w, 0) = 0$$

$$(FM' - 2): M(x, y, z, w, t) = 1, \forall t > 0,$$

Only when the three simplex  $\langle x, y, z, w, \rangle$  degenerate

$$(FM' - 3): M(x, y, z, w, t) = M(x, w, z, y, t) = M(z, w, x, y, t) = \dots$$

$$(FM' - 4):$$

$$M(x, y, z, w, t_1 + t_2 + t_3) \geq M(x, w, z, u, t_1) *$$

$$M(x, y, z, w, t_2) * M(x, y, z, w, t_3) * M(x, y, z, w, t_4)$$

$$(FM' - 5): M(x, y, z, w, \cdot): [0,1] \rightarrow [0,1] \text{ is left continuous,}$$

$$\forall x, y, z, u, w \in X, t_1, t_2, t_3, t_4 > 0$$

**Definition 2.13**

Let  $(X, M, *)$  be a fuzzy 3-metric space:

(1) A sequence  $\{x_n\}$  in fuzzy 3-metric space  $X$  is said to be convergent to a point  $x \in X$ , if

$$\lim_{n \rightarrow \infty} M(x_n, x, a, b, t) = 1 \text{ for all } a, b \in X \text{ and } t > 0$$

(2) A sequence  $\{x_n\}$  in fuzzy 3-metric space  $X$  is called a Cauchy sequence, if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, a, b, t) = 1 \text{ for all } a, b \in X \text{ and } t, p > 0$$

(3) A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

**Definition 2.14**

A function  $M$  is continuous is fuzzy 3-metric space if

$$x_n \rightarrow x, y_n \rightarrow y \text{ then } \lim_{n \rightarrow \infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t), \forall a, b \in X \text{ and } T > 0$$

**Definition 2.15**

Two mappings  $A$  and  $S$  on fuzzy 3-metric space  $X$  are weakly commuting iff,

$$M(ASu, SAu, a, b, t) \geq M(Au, Su, a, t) \forall u, a, b \in X \text{ and } t > 0$$

**Definition 2.16**

Two self maps  $f$  and  $g$  of a set  $X$  are occasionally weakly compatible (o.w.c.) iff there is a point  $x$  in  $X$  which is a coincidence point of  $f$  and  $g$  at which  $f$  and  $g$  commute.

**Some Basic Results**

**Lemma 2.1**

[15] for all  $x, y \in X, M(x, y)$  is non-decreasing.

**Lemma 2.2**

[7] Let  $\{y_n\}$  be a sequence in a fuzzy metric space  $(X, M, *)$  with the condition (FM-6) If there exists a number  $q \in (0, 1)$  such that

$$M(y_{n+2}, y_{n+1}, qt) \geq M(y_{n+1}, y_n, t), \quad \forall t > 0 \text{ and } n = 1, 2, 3, \dots$$

**Lemma 2.3**

[24] If for all  $x, y \in X, t > 0$  and for a number  $q \in (0, 1)$ ,

$$M(x, y, qt) \geq M(x, y, t), \text{ then } x = y$$

**Lemma 2.4**

Let  $(X, M, *)$  be a Fuzzy metric space,  $A$  and  $B$  are occasionally weakly compatible self maps of  $X$ . If  $A$  and  $B$  have a unique point of coincidence,  $w = Ax = Bx$ , then  $w$  is the unique common fixed point of  $A$  and  $B$ .

**Proof**

Since  $A$  and  $B$  are occasionally weakly compatible, there exists a point  $x \in X$  such that  $Ax = Bx = w$  and  $ABx = BAx$ . Thus,  $AAx = ABx = BAx$ , which says that  $Ax$  is also a point of coincidence of  $A$  and  $B$ . Since the point of coincidence  $w = Ax$  is unique by hypothesis,  $BAx = AAx = Ax$ , and  $w = Ax$  is a common fixed point of  $A$  and  $B$ . Moreover, if  $z$  is any common fixed point of  $A$  and  $B$ , then  $z = Az = Bz = w$  by the uniqueness of the point of coincidence.

Fisher [10] proved the following theorem for three mappings in complete metric space:

**Theorem 2.5**

Let  $S$  and  $T$  be continuous mappings of a complete metric space  $(X, d)$  into itself. Then  $S$  and  $T$  have a common fixed point in  $X$  iff there exists a continuous mapping  $A$  of  $X$  into  $S(X) \cap T(X)$ , which commutes with  $S$  and  $T$  and satisfy  $d(Ax, Ay) \geq \alpha d(Sx, Ty)$  for all  $x, y \in X$  and  $0 < \alpha < 1$ .

Then  $S, T$  and  $A$  have a unique common fixed point.

**Main Results**

**Theorem 3.1**

Let  $(X, M, *)$  be a complete fuzzy metric space with the condition (FM-6) and let  $P, Q, F$  and  $G$  be self mappings of  $X$ . Let pairs  $\{P, F\}$  and  $\{Q, G\}$  be occasionally weakly compatible If there exists  $h \in (0, 1)$  such that

$$(3.1.1)$$

$$M(Px, Qy, (ht)) \geq \phi \left( \left( \frac{M(Fx, Gy, t), M(Fx, Px, t), M(Qy, Gy, t)}{[M(Px, Gy, t) + M(Fx, Px, t)], [M(Qy, Fx, t) + M(Qy, Gy, t)]} \right) \right)$$

For all  $x, y \in X, \phi: [0, 1]^5 \rightarrow [0, 1]$  such that  $\phi(t, 1, 1, t, t) > t$  for all  $0 < t < 1$ , then there exists

a unique point  $w \in X$  such that  $Pw = Fw = w$  and a unique point

$X$  such that  $Qz = Gz = z$ . Moreover,  $z = w$ , so that there is a unique common fixed point of  $P, F, Q$  and  $G$ .

**Proof**

Let the pairs  $\{P, F\}$  and  $\{Q, G\}$  be occasionally weakly compatible, so there are points  $x, y \in X$  such that  $Px = Fx$  and  $Qy = Gy$ . We claim that  $Px = Qy$ . If not, by inequality (3.1.1), we have

$$M(Px, Qy, (ht)) \geq \phi \left( \left( \frac{M(Fx, Gy, t), M(Fx, Px, t), M(Qy, Gy, t)}{[M(Px, Gy, t) + M(Fx, Px, t)], [M(Qy, Fx, t) + M(Qy, Gy, t)]} \right) \right) \\ = \phi \left( \left( \frac{M(Fx, Gy, t), M(Fx, Px, t), M(Qy, Gy, t)}{M(Px, Px, t), M(Qy, Qy, t)} \right) \right)$$

$$\phi(M(Px, Qy, t), 1, 1, M(Px, Qy, t), M(Px, Qy, t)) > M(Px, Qy, t)$$

This is a contradiction, therefore  $Px = Qy$ , i.e.  $Px = Fx = Qy = Gy$ . Suppose that there is another point  $z$  such that  $Pz = Fz$  then by (3.1.1) we have  $Pz = Fz = Qy = Gy$ , so  $Px = Pz$  and  $w = Px = Fx$  is the unique point of coincidence of  $P$  and  $F$ . By Lemma (2.4)  $w$  is the only common fixed point of  $P$  and  $F$ . Similarly there is a unique point  $z \in X$  such that  $z = Qz = Gz$ . Assume that  $w \neq z$ . We have

$$M(w, z, ht) = M(Pw, Fz, ht) \\ \geq \phi \left( \left( \frac{M(Fw, Gz, t), M(Fw, Pw, t), M(Qz, Gz, t)}{[M(Pw, Gz, t) + M(Fw, Pw, t)], [M(Qz, Fw, t) + M(Qz, Gz, t)]} \right) \right) \\ \geq \phi \left( \left( \frac{M(w, z, t), M(w, w, t), M(z, z, t)}{[M(w, z, t) + M(w, w, t)], [M(z, w, t) + M(z, z, t)]} \right) \right) \\ = \phi(M(w, z, t)) = M(w, z, t)$$

Therefore we have  $z = w$ .

Hence  $z$  is a common fixed point of  $P, F, Q$  and  $G$ . The uniqueness of the fixed point holds from (3.1.1).

**Theorem 3.2**

Let  $(X, M, *)$  be a complete fuzzy 2-metric space and let  $S^r$  and  $T^r$  be continuous mappings of  $X$  in  $X$ , then  $S^r$  and  $T^r$  have a common fixed point in  $X$  if there exists continuous mapping  $A^r$  of  $X$  into  $S^r(X) \cap T^r(X)$  which weakly compatible with  $S^r$  and  $T^r$  and

$$M(A^r x, A^r y, q^r t, a) \geq \min \left\{ \frac{M(A^r y, A^r y, t, a), M(S^r x, A^r x, t, a), M(S^r x, T^r y, a, t)}{M(A^r y, A^r y, a, t), M(S^r x, A^r y, a, t)}, M(A^r x, T^r y, a, t), M(S^r x, A^r y, a, t) \right\} \dots (3.2.1)$$

for all  $x, y, a \in X, t > 0$ , and  $0 < q < 1$ .

And  $\lim_{n \rightarrow \infty} M(x, y, z, a, t) = 1, \forall x, y, z, a \in X \dots (3.2.2)$

Then  $S^r, T^r$  and  $A^r$  have a unique common fixed point.

**Proof**

We define a sequence  $\{x_n\}$  such that  $A^r x_{2n} = S^r x_{2n-1}$  and  $S^r x_{2n-1} = T^r x_{2n}, n = 1, 2, \dots$

We shall prove that  $\{A^r x_{2n}\}$  is a Cauchy sequence.

For this suppose  $x = x_{2n}$  and  $y = A^r x_{2n+1}$  in (3.2.1), we write

$$M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq$$

$$\min \left\{ \begin{aligned} &M(A^r x_{2n+1}, A^r x_{2n+1}, a, t), \frac{M(S^r x_{2n}, A^r x_{2n}, a, t)}{M(A^r x_{2n+1}, A^r x_{2n+1}, a, t)}, \\ &\frac{M(S^r x_{2n}, T^r x_{2n+1}, a, t)}{M(S^r x_{2n}, A^r x_{2n+1}, a, t)}, M(A^r x_{2n}, T^r x_{2n+1}, a, t) \\ &M(S^r x_{2n}, A^r x_{2n+1}, a, t) \end{aligned} \right\}$$

$$M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq$$

$$\min \left\{ \begin{aligned} &M(A^r x_{2n+1}, A^r x_{2n+1}, A^r, t), \frac{M(S^r x_{2n+1}, A^r x_{2n}, a, t)}{M(A^r x_{2n+1}, A^r x_{2n}, a, t)}, \\ &\frac{M(S^r x_{2n+1}, A^r x_{2n}, a, t)}{M(A^r x_{2n}, A^r x_{2n+1}, a, t)}, M(A^r x_{2n}, T^r x_{2n}, a, t) \\ &M(A^r x_{2n+1}, A^r x_{2n+1}, a, t) \end{aligned} \right\}$$

$$M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq M(A^r x_{2n+1}, A^r x_{2n}, a, q^r t)$$

$$=$$

$$\min \left\{ M(A^r x_{2n}, A^r x_{2n+1}, a, t), \frac{M(A^r x_{2n+1}, A^r x_{2n}, a, t)}{M(A^r x_{2n+1}, A^r x_{2n}, a, t)}, \frac{M(A^r x_{2n+1}, A^r x_{2n}, a, t)}{1} \right\}$$

$$\geq$$

$$\min \left\{ M(A^r x_{2n-1}, A^r x_{2n}, a, t/qr), M(A^r x_{2n}, A^r x_{2n-1}, a, t/qr), M(A^r x_{2n+1}, A^r x_{2n}, a, t/qr), M(A^r x_{2n}, A^r x_{2n+1}, a, t/qr) \right\}$$

$$1, 1, 1$$

Therefore

$$M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq M(A^r x_{2n}, A^r x_{2n+1}, a, t/q^r)$$

By induction

$$M(A^r x_{2k}, A^r x_{2m+1}, a, q^r t) \geq M(A^r x_{2m}, A^r x_{2k+1}, a, t/q^r)$$

For every  $k$  and  $m$  in  $N$ , Further if  $2m + 1 > 2k$ , then

$$M(A^r x_{2k}, A^r x_{2m+1}, a, q^r t) \geq M(A^r x_{2m}, A^r x_{2k+1}, a, \frac{t}{q^r}) \geq$$

$$\dots \dots \dots$$

$$\dots \dots \dots \geq$$

$$M(A^r x_0, A^r x_{2m+1}, a, \frac{t}{q^{2kr}}) \geq \dots \dots \dots (3.2.3)$$

If  $2k > 2m + 1$ , then

$$M(A^r x_{2k}, A^r x_{2m+1}, a, q^r t) \geq M(A^r x_{2k+1}, A^r x_{2m}, a, \frac{t}{q^r}) \geq$$

$$\dots \dots \dots$$

$$\geq M(A^r x_{2k-(2m-1)}, A^r x_0, a, \frac{t}{q^{(2m-1)r}}) \dots \dots \dots (3.2.4)$$

By simple induction with (3.2.3) and (3.2.4) we have

$$M(A^r x_n, A^r x_{n+p}, q^r t) \geq M(A^r x_0, A^r x_p, t/q^{nr})$$

For  $n = 2k, p = 2m + 1$  or  $n = 2k + 1, p = 2m + 1$

and by (FM-4)

$$M(A^r x_n, A^r x_{n+p}, a, q^r t) \geq$$

$$\min \left\{ M(A^r x_0, A^r x_1, a, \frac{t}{2q^{nr}}), M(A^r x_1, A^r x_p, a, \frac{t}{q^{nr}}) \right\} \dots \dots \dots (3.2.5)$$

If  $n = 2k, p = 2m$  or  $n = 2k + 1, p = 2m$

For every positive integer  $p$  and  $n$  in  $N$ , therefore

$$(A^r x_0, A^r x_p, a, \frac{t}{q^{nr}}) \rightarrow 1 \text{ as } n \rightarrow \infty$$

Thus  $\{A^r x_n\}$  is a Cauchy sequence. Since the space  $X$  is complete there exists  $z \in X$ , such that

$$\lim_{n \rightarrow \infty} A^r x_n = \lim_{n \rightarrow \infty} S^r x_{2n+1} = \lim_{n \rightarrow \infty} T^r x_{2n} = z$$

It follows that  $A^r z = S^r z = T^r z$  and therefore by weak compatibility

$$M(A^r z, A^{2r} z, a, q^r t) \geq$$

$$\min \left\{ M(T^r A^r z, A^r, A^r z, a, t), \frac{M(S^r z, A^r z, a, t)}{M(A^r A^r z, T^r A^r z, a, t)}, \frac{M(S^r z, T^r A^r z, a, t)}{M(S^r z, A^r A^r z, a, t)}, \right.$$

$$M(A^r z, T^r A^r z, a, t), M(S^r z, A^r A^r z, a, t)$$

$$M(A^r z, A^{2r} z, a, q^r t) \geq M(A^r z, T^r A^r z, a, t)$$

$$\geq M(S^r z, A^r T^r z, a, t) \geq M(A^r z, A^{2r} z, a, t) \dots \dots \dots \geq$$

$$M(A^r z, A^{2r} z, a, \frac{t}{q^{nr}})$$

since  $\lim_{n \rightarrow \infty} M(A^r z, A^{2r} z, a, \frac{t}{q^{nr}}) = 1 \Rightarrow A^r z = A^{2r} z$

Thus  $z$  is common fixed point of  $A, S$  and  $T$ .

For **uniqueness**, let  $w (w \neq z)$  be another

common fixed point of  $S^r, T^r$  and  $A^r$  for all  $r > 0$ . By

inequality we write

$$M(A^r z, A^r w, a, q^r t) \geq$$

$$\min \left\{ M(T^r w, A^r w, a, t), \frac{M(S^r z, A^r z, a, t)}{M(A^r w, T^r w, a, t)}, \frac{M(S^r z, T^r w, a, t)}{M(S^r z, A^r w, a, t)}, \right.$$

$$M(A^r z, T^r w, a, t), M(S^r z, A^r w, a, t)$$

$$M(A^r z, A^r w, a, q^r t) \geq$$

$$\min \left\{ M(w, w, a, t), \frac{M(z, z, a, t)}{M(w, w, a, t)}, \frac{M(z, w, a, t)}{M(z, w, a, t)} \right\}$$

$$M(z, w, a, t), M(z, w, a, t)$$

$S^r$  and  $T^r \min\{M(z, w, a, t)\}$

This implies that  $M(z, w, a, q^r t) \geq \min\{M(z, w, a, t)\}$

Therefore by lemma (2.3), we get  $z = w$ .

Now we are generalizing above theorems in complete fuzzy 3-metric space.

**Theorem 3.3**

Let  $(X, M, *)$  be a complete fuzzy 3-metric space and let  $S^r$  and  $T^r$  be continuous mappings of  $X$  in  $X$ , then  $S^r$  and  $T^r$  have a common fixed point in  $X$  if  $\exists$  continuous mapping  $A^r$  of  $X$  into  $S^r(X) \cap T^r(X)$  which commute weakly with  $S^r$  and  $T^r$  and

$$M(A^r x, A^r y, q^r t, a, b) \geq$$

$$\min \left\{ M(T^r y, A^r y, a, b, t), M(S^r x, A^r y, a, b, t), \right.$$

$$M(S^r x, T^r y, a, b, t), M(A^r x, T^r y, a, b, t), M(S^r x, A^r y, a, b, t) \left. \right\}$$

$$\dots \dots \dots (3.3.1)$$

for all  $x, y, a, b \in X, t > 0$ , and  $0 < q < 1$ . Here  $(a * b) = \min(a, b)$

And  $\lim_{n \rightarrow \infty} M(x, y, z, a, b, t) = 1 \forall x, y, z, a \in X$

$$\dots \dots \dots (3.3.2)$$

Then  $S^r, T^r$  and  $A^r$  have a unique common fixed point.

**Proof**

We define a sequence  $\{x_n\}$  such that

$A^r x_{2n} = S^r x_{2n-1}$  and  $A^r x_{2n-1} = T^r x_{2n}, n = 1, 2, \dots$

We shall prove that  $\{A^r x_n\}$  is a Cauchy sequence. For this suppose  $x = x_{2n}$  and  $y = x_{2n+1}$  in (3.3.1), we write

$$M(A^r x_{2n}, A^r x_{2n+1}, a, b, q^r t) \geq$$

$$\left( M(T^r x_{2n+1}, A^r x_{2n+1}, a, b, q^r t), M(S^r x_{2n}, A^r x_{2n}, a, b, t), \right.$$

$$\min \left\{ M(S^r x_{2n}, T^r x_{2n+1}, a, b, t), M(A^r x_{2n}, T^r x_{2n+1}, a, b, t), \right.$$

$$M(S^r x_{2n}, A^r x_{2n+1}, a, b, t) \left. \right\}$$

$$M(A^r x_{2n}, A^r x_{2n+1}, a, b, q^r t) \geq$$

$$\left( M(A^r x_{2n}, A^r x_{2n+1}, a, b, t), M(A^r x_{2n+1}, A^r x_{2n}, a, b, t), \right.$$

$$\min \left\{ M(A^r x_{2n+1}, T^r x_{2n}, a, b, t), M(A^r x_{2n}, T^r x_{2n}, a, b, t), \right.$$

$$M(A^r x_{2n+1}, A^r x_{2n+1}, a, b, t) \left. \right\}$$

$$M(A^r x_{2n}, A^r x_{2n+1}, a, b, q^r t) \geq$$

$$M(A^r x_{2n-1}, A^r x_{2n}, a, b, q^r t)$$

$$= \min \left\{ \begin{array}{l} M(A^r x_{2n}, A^r x_{2n+1}, a, b, t), M(A^r x_{2n+1}, A^r x_{2n}, a, b, t), \\ M(A^r x_{2n+1}, A^r x_{2n}, a, b, t), 1, 1 \end{array} \right\}$$

$$\geq \min \left\{ \begin{array}{l} M(A^r x_{2n-1}, A^r x_{2n}, a, b, t/q^r), M(A^r x_{2n}, A^r x_{2n-1}, a, b, t/q^r), \\ M(A^r x_{2n}, A^r x_{2n-1}, a, b, t/q^r), 1, 1 \end{array} \right\}$$

There fore  
 $M(A^r x_{2n}, A^r x_{2n+1}, a, b, q^r t) \geq M(A^r x_{2n-1}, A^r x_{2n}, a, b, t/q^r)$   
 By induction  $M(A^r x_{2k}, A^r x_{2m+1}, a, b, q^r t) \geq M(A^r x_{2m}, A^r x_{2k-1}, a, b, t/q^r)$   
 For every  $k$  and  $m$  in  $N$ , Further if  $2m + 1 > 2k$ , then  
 $M(A^r x_{2k}, A^r x_{2m+1}, a, b, q^r t) \geq M(A^r x_{2m}, A^r x_{2k-1}, a, b, \frac{t}{q^r}) \geq \dots \dots$   
 $\dots \dots \dots \geq M(A^r x_0, A^r x_{2m+1-2k}, a, b, t/q^{2kr})$   
 .....(3.3.3)  
 If  $2k > 2m + 1$ , then  
 $M(A^r x_{2k}, A^r x_{2m+1}, a, b, q^r t) \geq M(A^r x_{2k-1}, A^r x_{2m}, a, b, \frac{t}{q^r}) \geq \dots \dots$   
 $\dots \dots \dots \geq M(A^r x_{2k-(2m+1)}, A^r x_0, a, b, t/q^{(2m+1)r})$   
 .....(3.3.4)

By simple induction with (3.3.3) and (3.3.4) we have  
 $M(A^r x_n, A^r x_{n+p}, a, b, q^r t) \geq M(A^r x_0, A^r x_p, a, b, \frac{t}{q^r})$   
 For  $n = 2k, p = 2m + 1$  or  $n = 2k + 1, p = 2m + 1$  and by (FM-4)  
 $M(A^r x_n, A^r x_{n+p}, a, b, q^r t) \geq \left\{ M(A^r x_0, A^r x_1, a, b, \frac{t}{2q^{nr}}), M(A^r x_1, A^r x_p, a, b, \frac{t}{q^{nr}}) \right\}$   
 .....(3.3.5)

If  $n = 2k, p = 2m$  or  $n = 2k + 1, p = 2m$ .  
 For every positive integer  $p$  and  $n$  in  $N$ , by nothing that  
 $M(A^r x_0, A^r x_p, a, b, \frac{t}{q^{nr}}) \rightarrow 1$  as  $n \rightarrow \infty$   
 Thus  $\{A^r x_n\}$  is a Cauchy sequence. Since the space  $X$  is complete there exists  $z \in X$ , such that  
 $\lim_{n \rightarrow \infty} A^r x_n = \lim_{n \rightarrow \infty} S^r x_{2n+1} = \lim_{n \rightarrow \infty} T^r x_{2n} = z$   
 It follows that  $Az = Sz = Tz$  and so  
 $M(A^r z, A^{2r} z, a, b, qt) \geq \left\{ M(T^r A^r z, A^r A^r z, a, b, t), M(S^r z, A^r z, a, b, t), M(S^r z, T^r A^r z, a, b, t), \right.$   
 $\left. M(A^r z, T^r A^r z, a, b, t), M(S^r z, A^r A^r z, a, b, t), \right\}$   
 $M(A^r z, A^{2r} z, a, b, qt) \geq M(S^r z, T^r A^r z, a, b, t) \geq M(S^r z, A^r z, a, b, t) \geq M(A^r z, A^{2r} z, a, b, qt) \dots \dots \dots \geq M(A^r z, A^{2r} z, a, b, \frac{t}{q^{nr}})$   
 since  $\lim_{n \rightarrow \infty} M(A^r z, A^{2r} z, a, \frac{t}{q^{nr}}) = 1 \Rightarrow A^r z = A^{2r} z$ .

Thus  $z$  is common fixed point of  $A^r, S^r$  and  $T^r$ .

For uniqueness, let  $w (w \neq z)$  be another common fixed point of  $S^r, T^r$  and  $A^r$ . By (3.3.1) we write  
 $M(A^r z, A^r w, a, qt) \geq \min \left\{ \begin{array}{l} M(T^r w, A^r w, a, t), M(S^r z, A^r z, a, t), \\ M(S^r z, T^r w, a, t), M(A^r z, T^r w, a, t), M(S^r z, A^r w, a, t) \end{array} \right\}$   
 $M(A^r z, A^r w, a, qt) \geq \min \left\{ \begin{array}{l} M(w, w, a, t), M(z, z, a, t), \\ M(z, w, a, t), M(w, z, a, t), M(z, w, a, t) \end{array} \right\}$   
 $M(A^r z, A^r w, a, qt) \geq \min \{M(z, w, a, t)\}$

This implies that  
 $M(z, w, a, b, qt) \geq \min \{M(z, w, a, b, t)\}$   
 Therefore by lemma (2.3) we get  
 $z = w$ .

This completes the proof of the theorem.

**Conclusion**  
 Above established theorems are the generalization of the earlier results of fisher [10] and all others, also we extend this result is to fuzzy 2 and 3-metric spaces by obtaining some common fixed point theorems on fuzzy metric spaces.

**References**

1. Badard R., "Fixed point theorems for fuzzy numbers", *Fuzzy sets and systems* 13(3) (1984), 291-302.
2. Bose R. K. and Sahani D., "Fuzzy mappings and fixed point theorems", *Fuzzy Sets and Systems*, 21 (1987), 53-58.
3. Butnariu D , "Fixed points for Fuzzy mappings", *Fuzzy Sets & Systems*, 7(1982), 191-207.
4. Chang S. S. , "Fixed point theorems for fuzzy mappings", *Fuzzy Sets and Systems* 17 (1985), 181-187.
5. Chang S. S., Cho Y. J. , Lee B. S. and Lee G. M. , "Fixed degree and fixed point theorems for fuzzy mappings in probabilistic metric spaces", *Fuzzy Sets and Systems* 87(3) (1997), 325-334.
6. Chang S. S., Cho Y. J., Lee B. S. , Jung J. S. and Kang S. M. , "Coincidence point and minimization theorems in fuzzy metric spaces", *Fuzzy Sets and Systems* 88(1) (1997), 119-128.
7. Cho Y. J., "Fixed points and fuzzy metric spaces", *J. Fuzzy Math.* 5 no. 4 (1997), 949-962.
8. Deng Z. , "Fuzzy pseudo-metric space", *J. Math. Anal. Appl.* 86 (1982), 74-95.
9. Fang J. X., "On fixed point theorems in fuzzy metric spaces", *Fuzzy Sets and Systems* 46 (1979), 107-113.
10. Fisher B., "Mappings with a common fixed point", *Math. Seminar notes Kobe university* 7 (1979), 81-84.
11. Gahler S. , "Linear 2-normierte Raume", *Math. Nachr.* 28 (1964), 1-43.
12. Gahler S. , "Uber 2-Banach Raume", *Math. Nachr.* 42 (1969), 335-347.
13. Gahler S. , "2-metrische Raume and ihre topologische struktur", *Math. Nachr.* 26 (1983), 115-148.
14. George A. and Veeramani P. , "On some results in fuzzy metric spaces", *Fuzzy Sets and Systems* 64(3) (1994), 395-399.
15. Grabiec M. , "Fixed points in fuzzy metric space", *Fuzzy Sets and Systems* 27(3) (1988), 385-389.
16. Hadzic O., "Fixed point theorems for multi-valued mappings in some classes of fuzzy metric space", *Fuzzy Sets and Systems* 29 (1989), 115-125.
17. Heilpern S., "Fuzzy mappings and fixed point theorems", *J. Math. Anal. Appl.* 83 (1981), 566-569.
18. Jung J. S., Cho Y. J. and Kim J. K., "Minimization theorems for fixed point theorems in fuzzy

- metric spaces and applications*", *Fuzzy Sets and Systems* 61 (1994), 199-207.
19. Jung J. S. , Cho Y. J. , Chang S. S. and Kang S. M. , "Coincidence theorems for set valued mappings and Eklund's variational principle in fuzzy metric spaces", *Fuzzy Sets and Systems* 79 (1996), 239-250.
  20. Kaleva O. and Seikkala S. , "On Fuzzy metric spaces", *Fuzzy Sets and Systems* 12 (1984), 215-229.
  21. Kramosil I. and Michalek J. , "Fuzzy metric and statistical metric spaces", *Kybernetika* 11 (1975), 336-344.
  22. Kutukcu S. , Sharma S. and Tokgoz H. , "A Fixed point theorem in fuzzy metric spaces", *Int. Journal of Math. Analysis* 1 no.18 (2007), 861-872.
  23. Lee B. S., Cho Y. J. and Jung J. S., "Fixed point theorems for fuzzy mappings and applications", *Comm. Korean Math. Sci.* 11 (1996), 89-108.
  24. Mishra S. N., Sharma N. and Singh S. L., "Common fixed points of maps on fuzzy metric spaces", *Internet. J. Math. & Math Sci.* 17 (1994), 253-258.
  25. Schweizer B. and Sklar A. , "Statistical metric spaces", *Pacific Journal Math.* 10 (1960), 313-334.
  26. Sharma P. L, Sharma B. K. and Iseki K. , "Contractive type mapping on 2-metric spaces", *Math. Japonica* 21 (1976), 67-70.
  27. Sharma P. L. and Rajput S. S., "Fixed point theorem in Banach space", *Vikram Mathematical Journal* 4 (1983), 35-38.
  28. Som T. , "Some results on common fixed point in fuzzy Metric spaces", *Soochow Journal of Mathematics* 33(4) (2007), 553-561.
  29. Garg, Surendra, Shukla Manoj Kumar "Fixed Point Results For Weak Compatible Mapping In Fuzzy 2-Metric Space". *IARJSECT*, Vol.-3 Special Issue 4 pp 15-17 (2016).
  30. Garg, Surendra, Shukla Manoj Kumar "Fixed Point Results in fuzzy Metric Space". *Bull. Cal. Math. Soc.* 109(3) pp171-182 (2017).
  31. Turkoglu D and Rhodes B. E., "A fixed fuzzy point for fuzzy mapping in complete metric spaces", *Mathematical Communications* 10 (2005), 115-121.
  32. Yadava R. N. , Rajput S. S. , Choudhary S. and Bhardwaj R. K. , "Some fixed point theorems for rational inequality in 2- metric spaces", *Acta Cienc. Indica* 33(3) (2007), 709-714.
  33. Yadava R. N. , Rajput S. S. , Choudhary S. and Bhardwaj R. K. "Some fixed point theorems for non contraction type mapping on 2- Banach spaces", *Acta Cienc. Indica* 33 (2007), 737-744.
  34. Zadeh L. A., "Fuzzy Sets", *Information and control* 8 (1965), 338-353.
  35. Zadeh L. A., "Probability measures of fuzzy events", *J Math. Anal Appl.* 23(1968), 421-427.