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Generalization of Semiperfect Rings

Abstract

Any right R-module M is called a CS-module if every submodule of M is essential in a direct summand of M. A ring is said to be CS-ring if R as right R-module is CS[9].In this paper we study semiperfect ring in which each simple right R-module is essential in a direct summand of R. We call such ring as a extending for simple R-module. Here we find that for such rings, every simple R-module is weakly-injective if and only R is weakly –injective if and only if R is self-injective if and only if R is weakly-semisimple. Examples are constructed for which simple R-module is essential in a direct summand.

Keywords: **S**semiperfect Ring, CS-Module, Extending For Simple, Weakly–Injective, Weakly-Semisimple Ring, Self-Injective Ring.

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Introduction

Throughout this paper, unless otherwise stated, all rings have unity and all modules are right until. For any two right R-modules M and N, a submodule S of M is said to be essential in M denoted by $S \subset_e M$, if for any non-zero submodule L of M, $S \cap L \neq 0$. R is said to be semiperfect if it has a complete set $\{e_i\}_{i=1}^n$ of primitive orthogonal idempotent such that each $e_i \mathrm{Re}_i$ is a local ring.

J or Rad(R) will denote the Jacobson radical of R, Soc (M) will denote the socle of M. The injective hull of the right R-module M is denoted by E(M).The notations in this paper are standard and it may be found in [1] and [2].

Preliminaries

Definition 2.1

We say that M is extending for simple module if for each simple submodule S of M there is a direct summand M' of M such that S is essential in M' [5].

Definition 2.2

R is said to be extending for simple R-module if R as a right R-module is extending for simple R-module.

Definition 2.3

For any right R-module M, we take a direct decomposition $M = \sum \oplus M_i$. For a submodule N_i of M_i, we call $\sum \oplus N_i$ a standard submodule of M with respect to this decomposition $\sum \oplus M_i$. Thus a standard submodule means a standard submodule with respect to decomposition into indecomposable modules. For any right R-module M, we note that J(M) and Soc(M) are always standard submodule with respect to any decompositions of M.

Definition 2.4

Let M and N be two right R-modules. We say that M is weakly N-injective if and only if every map $\phi : N \to E(M)$ from N into the injective hull E(M) of M may be written as a composition $\sigma \circ \hat{\phi}$, where $\hat{\phi} : N \to M$ and $\sigma : \mathrm{M} \to \mathrm{E(M)}$ is monomorphism. We say that M is weakly –injective if and only if it is weakly M-injective for every finitely generated module N.

Definition 2.5

A ring R is said to be right weakly-semisimple if every right R-module M is weakly-injective.

Lemma 2.6

Let S be any simple submodule of M which is essential in M then M is an indecomposable module.

Proof

Let $M = M_1 \oplus M_2$ where M_1 and M_2 are submodule of M. Given that S is essential in M. Therefore $S \cap M_1 \neq 0$ and $S \cap M_2 \neq 0$. Since S is simple implies that $S \subset M_1$ and $S \subset M_2$. This implies that $S \subset M_1 \cap M_2$ which is contradiction.

Lemma 2.7

If any right R-module M has essential simple submodules S, then Soc (M) = S.

Proof

Let L and S be two simple submodules of M. Since S is essential in M. Therefore $S \cap L \neq 0$, implies that $S \subset L$ or $L \subset S$ i.e. S = L. Hence Soc (M) = S.

Lemma 2.8

Let R be a semiperfect ring and let $e_1, e_2, \ldots\ldots\ldots, e_m$ be a basic set of primitive idempotents for R. If P_R is projective then there exist sets $A_1, A_2, \ldots\ldots\ldots, A_m$ (unique to within cardinality and possibly empty) such that

$$P \cong (e_1R)^{(A_1)} \oplus (e_2R)^{(A_2)} \oplus \ldots\ldots\ldots \oplus (e_kR)^{(A_k)}$$

Proof

See [1, Theorem 27.11, Page 306].

Lemma 2.9

Suppose that $K_1 \subset M_1 \subset M, K_2 \subset M_2 \subset M$ and $M_1 \oplus M_2$. Then $K_1 \oplus K_2 \subset_e M_1 \oplus M_2$ if and only if $K_1 \subset_e M_1$ and $K_2 \subset_e M_2$.

Proof

See [1, Proposition 5.20(2), Page 75].

Proposition 2.10

Let R be any semiperfect ring such that R_R is extending for simple submodule, then

(i) For any projective R-module P, Soc (P) is essential in P.

(ii) If Q is another projective R-module such that $Soc(Q) \cong Soc(P)$ then $Q \cong P$.

Proof

Since R is semiperfect, we may write $R = e_1R \oplus e_2R \oplus \ldots\ldots\ldots \oplus e_nR$, where $P = \{e_1R, e_2R, \ldots\ldots\ldots, e_kR\}$ $(k \leq n)$ an irredundant is complete set of representative for the projective indecomposable R-modules.Let $L = \{S_1, S_2, \ldots\ldots\ldots, S_k\}$ be an irredundant complete set of representatives for the simple R-modules.

Since R_R is extending for simple submodule hence for any simple submodule S_i there exist a direct summand eR or R such that S_i is essential in eR from Lemma 2.6, eR should be indecomposable R-module.

Therefore $eR \cong e_jR$ for some $j \in \{1,2, \ldots\ldots\ldots, k\}$.Thus we can define a function $f : L \to P$ by $f(S_i) = e_jR$, f must be one-one, hence onto. Also by Lemma 2.7, $Soc(e_iR) \cong S_i$ i.e. $Soc(e_iR) = S_i$ is the unique essential submodule of e_iR. Thus Soc (P) is essential in P as proved for indecomposable projective R-module $e_iR = P$.

Let P be an arbitrary projective R-module. Since R is semiperfect there exist sets $A_i, i = 1,2, \ldots\ldots\ldots, k$ such that

$$P \cong (e_1R)^{(A_1)} \oplus (e_2R)^{(A_2)} \oplus \ldots\ldots\ldots \oplus (e_kR)^{(A_k)}$$

By Lemma 2.8, since Soc (P) is an essential submodule of P. Therefore

$$Soc(P) \cong (Soc(e_1R))^{(A_1)} \oplus (Soc(e_2R))^{(A_2)} \oplus \ldots\ldots\ldots \oplus (Soc(e_kR))^{(A_k)}$$

Using Lemma 2.9, we get

$$(Soc(e_1R))^{(A_1)} \oplus (Soc(e_2R))^{(A_2)} \oplus \ldots\ldots\ldots \oplus (Soc(e_kR))^{(A_k)} \subset_e (e_1R)^{(A_1)} \oplus (e_2R)^{(A_2)} \oplus \ldots\ldots\ldots \oplus (e_kR)^{(A_k)}$$

i.e. $Soc(P) \subset_e P$.

Let $Q = (e_1R)^{(B_1)} \oplus (e_2R)^{(B_2)} \oplus \ldots\ldots\ldots \oplus (e_kR)^{(B_k)}$ be any other projective R-module such that $Soc(Q) \cong Soc(P)$.

Then

$$(Soc(e_1R))^{(B_1)} \oplus (Soc(e_2R))^{(bA_2)} \oplus \ldots\ldots\ldots \oplus (Soc(e_kR))^{(B_k)} \cong (Soc(e_1R))^{(A_1)} \oplus (Soc(e_2R))^{(A_2)} \oplus \ldots\ldots\ldots \oplus (Soc(e_kR))^{(A_k)}$$

and so by the Krull-Schmidt theorem there is a bijection A_i *and* B_i for i = 1, 2,…………,k. Therefore

$$(e_1R)^{(B_1)} \oplus (e_2R)^{(B_2)} \oplus \ldots\ldots\ldots \oplus (e_kR)^{(B_k)} \cong (e_1R)^{(A_1)} \oplus (e_2R)^{(A_2)} \oplus \ldots\ldots\ldots \oplus (e_kR)^{(A_k)}$$

i.e. $Q \cong P$.

Proposition 2.11

If R is semiperfect and extending for simple right R-module then R is left perfect.

Proof

We shall show that each cyclic R-module has non-zero socle [4, Lemma 9].For any cyclic R-module xR, if it is contained in e_iR then since e_iR

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Asian Resonance

has essential simple submodule S_i. Therefore $S_i \cap xR \neq 0$. Thus $S_i \subset xR$ i.e. $Soc(xR) \neq 0$.

On the other hand if xR contains any $e_i R$ then obviously $Soc(e_i R) \subset Soc(xR)$ i.e. $Soc(xR) \neq 0.$

Theorem 2.12

Let R be any semiperfect and extending for simple right R-module, then following conditions are equivalent:

(i) Every right simple R-module is weakly-injective.
(ii) R is weakly-injective.
(iii) R is self-injective ring.
(iv) R is weakly-semisimple ring.
(v) Every right R-module is weakly-injective.

Proof

$(i) \Rightarrow (ii)$ Let $R = e_1 R \oplus e_2 R \oplus .... \oplus e_k R \, (k \leq n)$ and $S = \{S_1, S_2, ..., S_k\}$. Given that S_i is weakly-injective and S_i is essential in $e_i R$ as R is extending. Therefore $e_i R$ is weakly-injective. Also finite direct sum of weakly-injective is weakly-injective. Therefore $R = e_1 R \oplus e_2 R \oplus .... \oplus e_k R$ is weakly-injective.

$(ii) \Rightarrow (iii)$ Suppose R is weakly-injective. By Proposition 2.11, R is left perfect. Over left perfect ring R, R is weakly-injective if R is self injective [6, Lemma 2.8].

$(iii) \Rightarrow (iv)$ Given that R is self-injective hence it would be weakly-injective. Every direct summand of R is injective and hence every direct summand of R is weakly-injective. Therefore R is weakly-semisimple ring [7, Theorem 2.4].

$(iv) \Rightarrow (v)$ Since R is weakly-semisimple, therefore every right R-module M will be weakly-injective.

$(v) \Rightarrow (i)$ Obvious.

Example 2.13

(1) Let $R = \begin{bmatrix} Z & Q \\ 0 & Q \end{bmatrix}$ is a weakly R-injective. Here simple R-module $[0, Q] = e_{22} R$ is not weakly R-injective i.e. R is not weakly-semisimple ring and R is also not self-injective ring.

(2) For a Boolean ring R, following are equivalent -
(i) R is weakly R-injective.
(ii) R is weakly-semisimple ring.
(iii) R is self-injective ring.

Proof

For any Boolean ring, its injective hull E(R) and classical quotient ring Q(R) of R are same i.e. R = E(R) = Q(R).

Example 2.14

Let $S = \begin{bmatrix} B & A \\ 0 & A \end{bmatrix}$ where $A = Q(x_1, x_2, ...x_n)$ a field of rational functions in n independents and $B = (x_1^2, x_2^2, ...x_n^2)$ is a subfield of A. Let $f : A \to B$ defined by $f(x_i) = x_i^2, \, f(a) = a \, \forall a \in Q, \forall i = 1,2, ..., n$ then B is epimophic image of A [2, page 338]

or $S = \dfrac{Z}{p^2 Z}$

S has three right ideals S, J = Rad(S) = xS and (0).

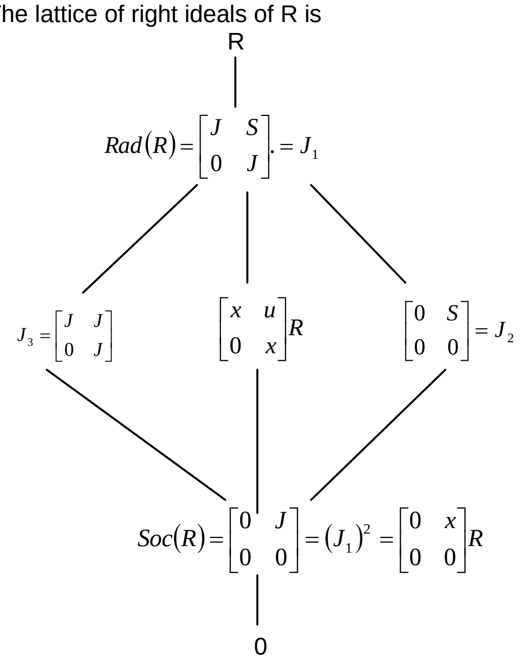
Also $J^2 \subset J.$

Therefore $J^2 = 0.$

Now let $R = \left\{ \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \mid a, t \in S \right\} \subset \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$

i.e. R is the split extension of the ring S[3].

The lattice of right ideals of R is



where $u \notin J$ in the generator $\begin{bmatrix} x & u \\ 0 & x \end{bmatrix}$ for the cyclic R-module $\begin{bmatrix} x & u \\ 0 & x \end{bmatrix} R$.

Since $End(R_R) \cong R$ is a local ring hence R_R is indecomposable and it is semiperfect. The irredundant complete set of representatives for projective indecomposable R-module contains single element namely R only; and hence irredundant complete set of representatives for simple R-module also contains only single element namely $Soc(R) = \begin{bmatrix} 0 & J \\ 0 & 0 \end{bmatrix}$.

Clearly $\frac{R}{Rad(R)} = Soc(R) \subset_e R$ i.e. R is semiperfect and R_R is extending for simple R-module. However the factors ring $\overline{R} = \frac{R}{Soc(R)}$ is also semiperfect but not extending for simple module as $\frac{J_3}{Soc(R)}, \frac{K_u}{Soc(R)}, \frac{J_2}{Soc(R)}$ are three simple $\overline{R}$-modules. Clearly intersection of any two is zero i.e. $\overline{R}$ is not extending for the simple $\overline{R}$-modules $\frac{K_u}{Soc(R)}$.

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