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Application of Runge-Kutta Fourth Order (RK-4) Method to Solve Logistic Differential Equations



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Abstract

In this paper, Runge-Kutta fourth order (RK-4) method is employed to obtain approximate solution of Logistic differential equations which are first order non-linear differential equations used in to model the growth of populations. The results show that method converges rapidly and approximates the exact solution very accurately.

Keywords: Runge-Kutta Fourth Order Method, C-language, Logistic Differential Equation, Stable and Unstable Problem.

Introduction

Belgian Mathematician and Sociologist Pierre Francois Verhulst¹ was introduced Logistic differential equation to model the growth of populations limited by finite resources. The Logistic differential equation is given by

$$\frac{dP}{dt} = rP \left[1 - \frac{P}{K} \right] \dots \dots \dots (1)$$

Here $P(t)$ is called the population size at time t and $\frac{dP}{dt}$ gives the change in population size over time t . (1) contains two positive parameters namely r and K . The first parameter r is called the growth parameter and second parameter is called the carrying capacity. Solow² used Logistic differential equation to discussed a contribution to the theory of economic growth.

Runge-Kutta fourth order (RK-4) method was developed around 1900 by the German mathematicians C. Runge and M.W. Kutta. The RK-4 method is a method of order four, meaning that the total accumulated error is on the order of $o(h^4)$ while the local truncation error is on the order of $o(h^5)$. A history of Runge-Kutta methods was given by Butcher³. Dormand and Prince⁴ gave a family of embedded Runge-formulae. Zingg and Chisholm⁵ discussed the Runge-Kutta method for linear ordinary differential equation. Milne⁶ gave a note on the Runge-Kutta method. Cash and Karp⁷ established a variable order Runge-Kutta method for initial value problems with rapidly varying right hand sides. Ralston⁸ gave Runge-Kutta method with minimum error bounds. An order bound for Runge-Kutta method was given by Butcher⁹. Bogacki and Shampine¹⁰ explained 3(2) pair of Runge-Kutta formulas. A modification of the Runge-Kutta fourth order method was given by Blum¹¹. Cash¹² used a class of implicit Runge-Kutta methods for numerical integration of stiff ordinary differential equations. Mehdi and Kareem¹³ solved Lü chaotic system using fourth order Runge-Kutta method. Yang and Sten¹⁴ applied Runge-Kutta method for solving uncertain differential equations. Enright and Muir¹⁵ used efficient classes of Runge-Kutta methods for two point boundary value problems. Application of the fourth order Runge-Kutta method for the solution of high-order general initial value problems was given by Cortell¹⁶. Yaakub and Evans¹⁷ established a fourth order Runge-Kutta RK(4,4) method with error control. Estimating the error of the classic Runge-Kutta formula introduced by Hosea and Shampine¹⁸. A simplified derivation and analysis of fourth order Runge-Kutta method was given by Musa et.al.¹⁹.

This paper uses Runge-Kutta fourth order(RK-4) method to solve Logistic differential equations. The advantage of this proposed method is its capability for obtaining exact solution without any difficulty and spending a very little time. The aim of this work is to establish exact solution or approximate solution of high degree of accuracy for Logistic differential equations using Runge-Kutta fourth order(RK-4) method.

Runge-Kutta Fourth Order (RK-4) method for First Order I.V.P.

Consider the first order I.V.P. $\frac{dy}{dx} = f(x, y) \dots \dots (2)$ with $y(x_0) = y_0 \dots \dots \dots (3)$
 By Runge-Kutta fourth order(RK-4) method, the sequence of approximation for y is given by

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$x_{n+1} = x_n + h, \text{ for } n = 0, 1, 2, 3, \dots$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

Here h is the interval between equidistant values of x .

Logistic differential equation which is given by (1) with initial condition $P(t_0) = P_0$ can be treat as a first order initial value problem given by (2)&(3) and solved by the above discussed method.

Stability and Conditioning

If in an initial value problem, the small changes either in function f or in the initial condition induces large effects on the solution of the problem then the problem is said to be ill-conditioned or unstable. Conversely, a problem is said to be well-conditioned or stable if small changes in the data induces small changes in the corresponding solution of problem.

A solution $y(x)$ of initial value problem (2) with initial condition (3) is said to be stable with respect to the initial condition (3) if, given any $\epsilon > 0$, there is a $\delta > 0$ such that any other solution $\bar{y}(x)$ of (2) with initial condition (3) satisfying $|y(x) - \bar{y}(x)| \leq \epsilon$ whenever $|y(x_0) - \bar{y}(x_0)| \leq \delta$ for all $x > x_0$(4)

C-Program of Runge-Kutta Fourth Order (RK-4) Method for First Order Initial Value Problems

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
float f(float x, float y)
{return 0.08*y-0.00008*y*y+0*x;}
int main()
{ float x0,y0,h,k1,k2,k3,k4,y1,x;
int iter,i;
clrscr();
printf("Enter the value of x0 and y0 \n");
scanf("%f%f",&x0,&y0);
printf("Enter the value of h \n");
scanf("%f",&h);
printf("Enter the value of iteration");
scanf("%d",&iter) ;
for(i=1;i<=iter;i++)
{
x=x0+h;
k1=h*f(x0,y0);
k2=h*f(x0+h*0.5,y0+k1*0.5);
```

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```
k3=h*f(x0+h*0.5,y0+k2*0.5);
k4=h*f(x0+h,y0+k3);
y1=y0+0.16666*(k1+2*k2+2*k3+k4);
printf("the value of y=%f at x=%f\n",y1,x);
x0=x;
y0=y1;
}
getch();
return 0;
}
```

Applications

In this section, some applications are given in order to demonstrate the effectiveness of Runge-Kutta fourth order (RK-4) method to solve Logistic differential equations.

Application: 1

The Logistic differential equation(1) with growth parameter $r = 1$, carrying capacity $K = 10$ and $P(0) = 2$ is given by

$$\frac{dP}{dt} = P \left[1 - \frac{P}{10} \right] \dots \dots \dots (5)$$

with $P(0) = 2 \dots \dots \dots (6)$

Application: 2

The Logistic differential equation(1) with growth parameter $r = 1$, carrying capacity $K = 1$ and $P(0) = 5$ is given by

$$\frac{dP}{dt} = P[1 - P] \dots \dots \dots (7)$$

with $P(0) = 5 \dots \dots \dots (8)$

Application: 3

The Logistic differential equation(1) with growth parameter $r = 0.08$, carrying capacity $K = 1000$ and $P(0) = 100$ is given by

$$\frac{dP}{dt} = 0.08P \left[1 - \frac{P}{1000} \right] \dots \dots \dots (9)$$

with $P(0) = 100 \dots \dots \dots (10)$

Application: 4

The Logistic differential equation(1) with growth parameter $r = 0.25$, carrying capacity $K = 20$ and $P(0) = 1$ is given by

$$\frac{dP}{dt} = 0.25P \left[1 - \frac{P}{20} \right] \dots \dots \dots (11)$$

with $P(0) = 1 \dots \dots \dots (12)$

Output of Application: 1

```
Enter the value of x0 and y0
0
2
Enter the value of h
0.1
Enter the value of iteration6
the value of y=2.164800 at x=0.100000
the value of y=2.339209 at x=0.200000
the value of y=2.523145 at x=0.300000
the value of y=2.716415 at x=0.400000
the value of y=2.918711 at x=0.500000
the value of y=3.129598 at x=0.600000
```

Output of Application: 2

```

Enter the value of x0 and y0
0
5
Enter the value of h
0.1
Enter the value of iteration6
the value of y=3.621821 at x=0.100000
the value of y=2.898755 at x=0.200000
the value of y=2.455187 at x=0.300000
the value of y=2.156574 at x=0.400000
the value of y=1.942764 at x=0.500000
the value of y=1.782826 at x=0.600000
    
```

Output of Application: 3

```

Enter the value of x0 and y0
0
100
Enter the value of h
0.1
Enter the value of iteration6
the value of y=100.722282 at x=0.100000
the value of y=101.449188 at x=0.200000
the value of y=102.180748 at x=0.300000
the value of y=102.916977 at x=0.400000
the value of y=103.657898 at x=0.500000
the value of y=104.403534 at x=0.600000
    
```

Output of Application: 4

```

Enter the value of x0 and y0
0
1
Enter the value of h
0.1
Enter the value of iteration6
the value of y=1.024018 at x=0.100000
the value of y=1.048581 at x=0.200000
the value of y=1.073700 at x=0.300000
the value of y=1.099386 at x=0.400000
the value of y=1.125649 at x=0.500000
the value of y=1.152502 at x=0.600000
    
```

**Comparison between exact and RK-4 method solutions
Application: 1**

x	Exact Solution $y(x)$	RK-4 Method Solution $y(x)$
0.1	2.164807	2.164800
0.2	2.339223	2.339209
0.3	2.523167	2.523145
0.4	2.716446	2.716415
0.5	2.918751	2.918711
0.6	3.129649	3.129598

Application: 2

x	Exact Solution $y(x)$	RK-4 Method Solution $y(x)$
0.1	3.621482	3.621821
0.2	2.898421	2.898755
0.3	2.454919	2.455187
0.4	2.156362	2.156574
0.5	1.942594	1.942764
0.6	1.782688	1.782826

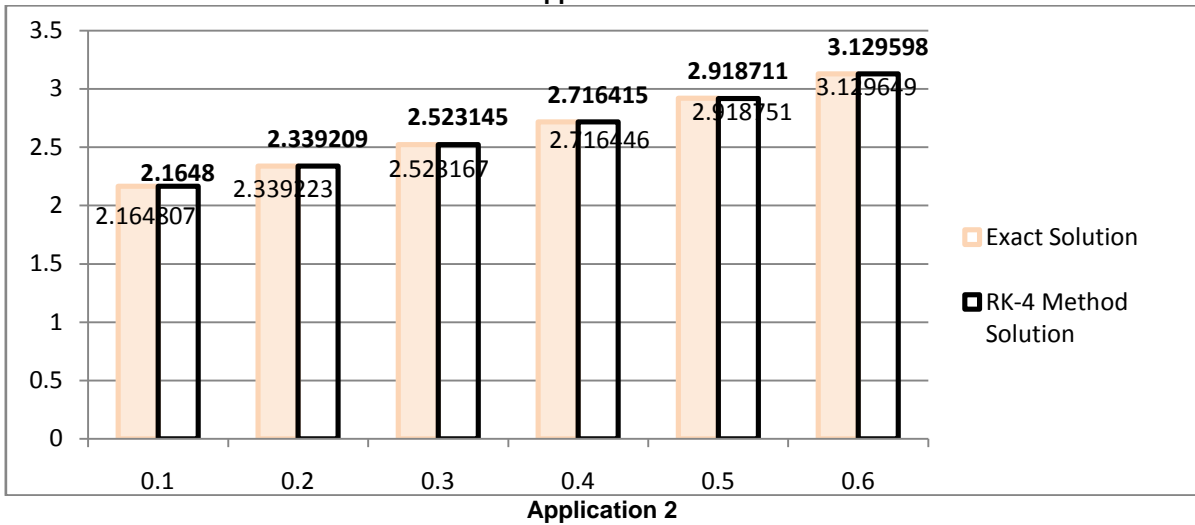
Application: 3

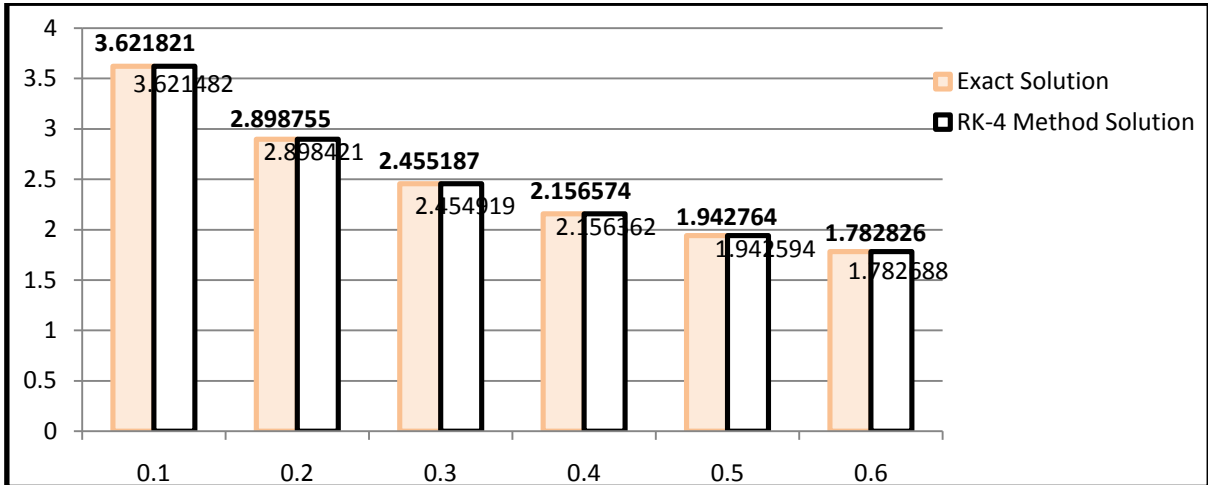
x	Exact Solution $y(x)$	RK-4 Method Solution $y(x)$
0.1	100.722308	100.722282
0.2	101.449244	101.449188
0.3	102.180831	102.180748
0.4	102.917089	102.916977
0.5	103.658041	103.657898
0.6	104.403706	104.403534

Application: 4

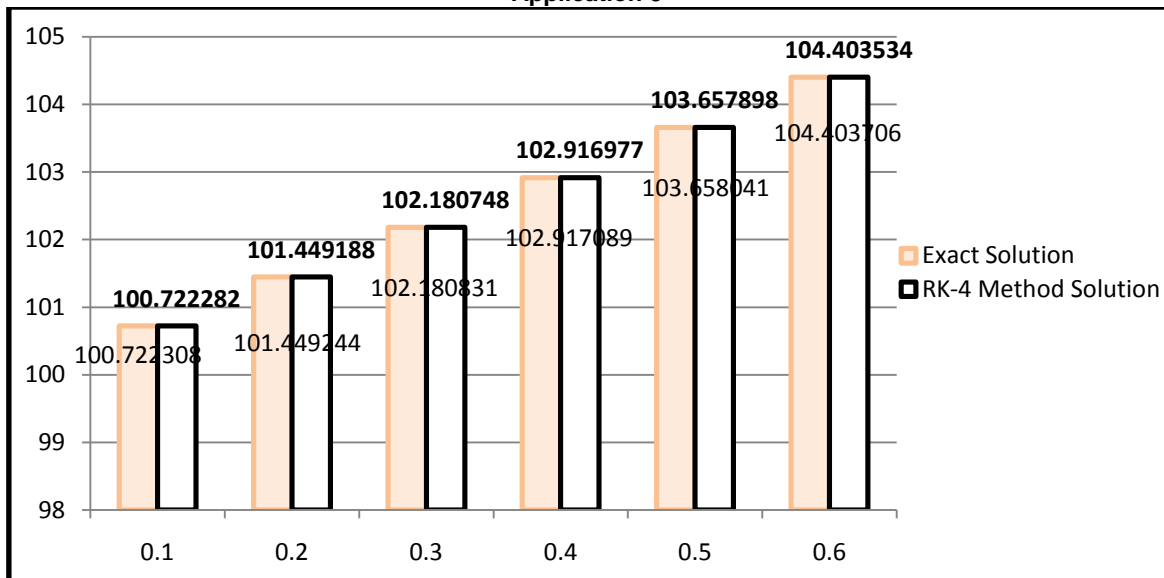
x	Exact Solution $y(x)$	RK-4 Method Solution $y(x)$
0.1	1.024019	1.024018
0.2	1.048583	1.048581
0.3	1.073703	1.073700
0.4	1.099389	1.099386
0.5	1.125654	1.125649
0.6	1.152508	1.152502

Comparison between Exact and RK-4 Method Solutions by Graphical Representation using above Data Application 1

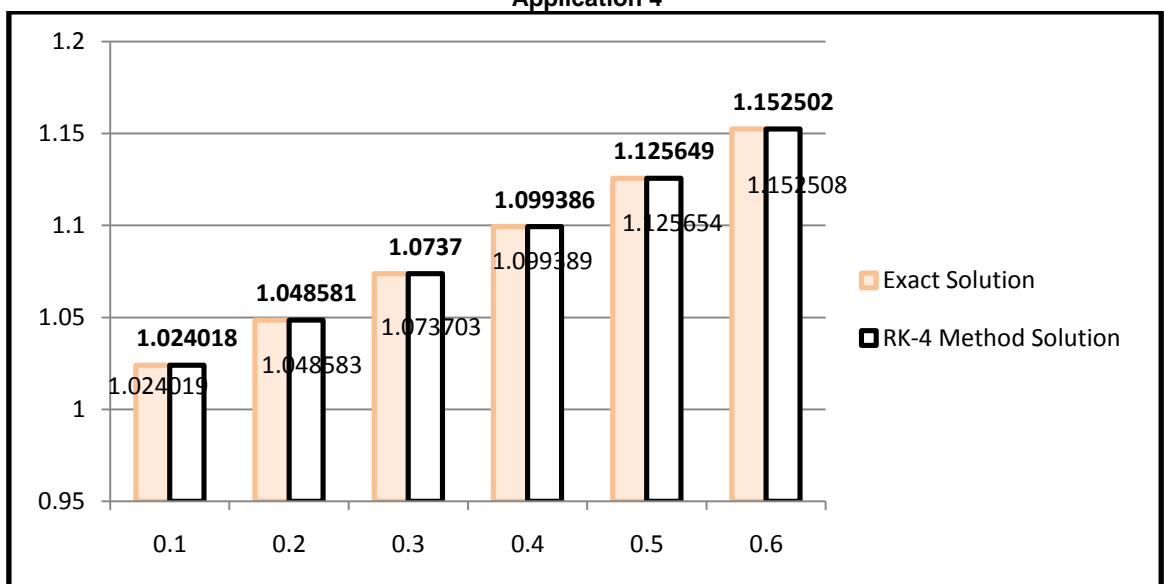




Application 3



Application 4



Conclusion

In this paper, we have successfully developed the Runge-Kutta fourth order(RK-4) method to solve the Logistic differential equations and comparison between exact and RK-4 method solutions are given in graphical and tabular form. The given applications show that the RK-4 method needless computational work to obtained solution of Logistic differential equations with high degree of accuracy.

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