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Solution of Linear Partial Integro-Differential Equations Using Mahgoub Transform

Abstract

In this paper, we used Mahgoub transform for solving linear partial integro-differential equations. The technique is described and illustrated with application. This technique gives the exact results using very less computational work.

Keywords: Linear Partial Integro-Differential Equation, Mahgoub Transform, Convolution Theorem, Inverse Mahgoub Transform.

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Introduction

Mathematical modeling of real life problems usually results in functional equations e.g. differential equations, partial differential equations, integral equations, integro-differential equations, stochastic equations, delay differential equations, partial integro-differential equations and others. In particular partial integro-differential equations arise in many scientific and engineering applications such as mathematical physics, visco-elasticity, finance, heat transfer, diffusion process, nuclear reactor dynamics, in general neutron diffusion, nano-hydrodynamics and fluid dynamics.

The general linear partial integro-differential equation is given by

$$\sum_{i=0}^{m} a_i \frac{\partial^i u(x,t)}{\partial t^i} + \sum_{i=0}^{n} b_i \frac{\partial^i u(x,t)}{\partial x^i} + cu + \sum_{i=0}^{r} d_i \int_0^t k_i\,(t,s) \frac{\partial^i u(x,s)}{\partial x^i} + f(x,t) = 0 \ldots \ldots (1)$$

(with prescribed conditions), where the kernels $k_i(t,s)$ and $f(x,t)$ are known functions and a_i, b_i, c and d_i are constants or functions of x.

The Mahgoub transform of the function $F(t)$is defined as [6]:

$$M\{F(t)\} = v \int_0^\infty F(t)e^{-vt}\,dt$$
$$= H(v), t \geq 0, k_1 \leq v \leq k_2$$

where M is Mahgoub transform operator.

The Mahgoub transform of the function $\mathrm{F(t)}$ for $\mathrm{t} \geq 0$ exist if $\mathrm{F(t)}$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Mahgoub transform of the function $\mathrm{F(t)}$.

Review of Literature

Appell et al. [1] discussed the partial integral operators and integro differential equations. Bahuguna and Dabas [2] gave existence and uniqueness of a solution to a PIDE by the method of lines. Yanik and Fairweather [3] used finite element methods for parabolic and hyperbolic partial integro-differential equations. Dehghan [4] discussed the solution of a partial integro-differential equation arising from viscoelasticity. Efficient solution of a partial integro-differential equation in finance was given by Sachs and Strauss [5]. Mahgoub [6] gave the new integral transform “Mahgoub Transform”.

Mahgoub and Alshikh [7] applied Mahgoub transform for solving partial differential equations. Fadhil [8] gave the convolution for Kamal and Mahgoub transforms. Taha et. al. [9] gave the dualities between Kamal & Mahgoub integral transforms and some famous integral transforms. For modeling biofluids flow in fractured biomaterials, Zadeh [10] gave an integro-partial differential equation. Thorwe and Bhalekar [11] used Laplace transform method for solving partial integro-differential equations. Mohand and Tarig [12] applied Elzaki transform method for solving partial integro-differential equations. Aboodh et al. [13] gave the solution of partial integro-differential equations by using Aboodh and double Aboodh transform methods. Mohand [15] used double Elzaki transform method for solving partial integro-differential equations. Aggarwal et al. [16] discussed a new application of Mahgoub transform for solving linear Volterra integral equations. Aggarwal et al. [17] solved the linear Volterra integro-differential equations of second kind using Mahgoub transform.

The object of the present study is to determine exact solutions for linear partial integro-differential equations using Mahgoub transform without large computational work.

Linearity Property of Mahgoub Transforms

If $M\{F(t)\} = H(v)$ and $M\{G(t)\} = I(v)$ then

$M\{aF(t) + bG(t)\} = aM\{F(t)\} + bM\{G(t)\}$
$= aH(v) + bI(v)$,where a, b are arbitrary constants.

Mahgoub Transform of Some Elementary Functions [6, 8]

S.N.	$F(t)$	$M\{F(t)\} = H(v)$
1.	1	1
2.	t	$\frac{1}{v}$
3.	t^2	$\frac{2!}{v^2}$
4.	$t^n, n \in N$	$\frac{n!}{v^n}$
5.	e^{at}	$\frac{v}{v-a}$
6.	$sinat$	$\frac{av}{v^2+a^2}$
7.	$cosat$	$\frac{v^2}{v^2+a^2}$
8.	$sinhat$	$\frac{av}{v^2-a^2}$
9.	$coshat$	$\frac{v^2}{v^2-a^2}$

Mahgoub transform of some partial derivatives of the function $u(x,t)$[7]

If $M\{u(x,t)\} = H(x,v)$ then

(a) $M\left\{\frac{\partial u(x,t)}{\partial t}\right\} = vH(x,v) - vu(x,0) \ldots..(3)$

(b) $M\{\frac{\partial^2 u(x,t)}{\partial t^2}\} = v^2 H(x,v) - v^2 u(x,0) - vu_t(x,0) \ldots \ldots.(4)$

(c) $M\left\{\frac{\partial^n u(x,t)}{\partial t^n}\right\} = v^n H(x,v) - v^n u(x,0) - v^{n-1}u_t(x,0) - \cdots \ldots - vu_{ttt \ldots.(n-1)times}(x,0) \ldots.(5)$

(d) $M\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \frac{dH(x,v)}{dx} \ldots \ldots \ldots (6)$

(e) $M\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} = \frac{d^2H(x,v)}{dx^2} \ldots \ldots..(7)$

(f) $M\left\{\frac{\partial^n u(x,t)}{\partial x^n}\right\} = \frac{d^nH(x,v)}{dx^n} \ldots \ldots..(8)$

Convolution of two functions [8]

Convolution of two functions $F(t)$ and $G(t)$ is denoted by $F(t) * G(t)$ and it is defined by

$$F(t) * G(t) = F * G = \int_0^t F(x)G(t-x)dx = \int_0^t F(t-x)G(x)dx$$

Convolution theorem for Mahgoub transforms [8]

If $M\{F(t)\} = H(v)$ and $K\{G(t)\} = I(v)$ then

$$M\{F(t) * G(t)\} = \frac{1}{v}M\{F(t)\}M\{G(t)\} = \frac{1}{v}H(v)I(v)$$

Inverse Mahgoub Transform

If $M\{F(t)\} = H(v)$ then $F(t)$ is called the inverse Mahgoub transform of $H(v)$ and mathematically it is defined as

$F(t) = M^{-1}\{H(v)\}$

where M^{-1} is the inverse Mahgoub transform operator.

Inverse Mahgoub transform of some elementary functions

S.N.	$H(v)$	$F(t) = M^{-1}\{H(v)\}$
1.	1	1
2.	$\frac{1}{v}$	t
3.	$\frac{1}{v^2}$	$\frac{t^2}{2!}$
4.	$\frac{1}{v^n}, n \in N$	$\frac{t^n}{n!}$
5.	$\frac{v}{v-a}$	e^{at}
6.	$\frac{v}{v^2+a^2}$	$\frac{sinat}{a}$
7.	$\frac{v^2}{v^2+a^2}$	$cosat$
8.	$\frac{v}{v^2-a^2}$	$\frac{sinhat}{a}$
9.	$\frac{v^2}{v^2-a^2}$	$coshat$

Mahgoub transform for linear partial integro-differential equations

In this section, we present Mahgoub transform for solving linear partial integro-differential equations given by (1). In this work, we will assume that the kernels $k_i(t,s)$ of (1) are difference kernel that can be expressed by difference $(t-s)$. The linear partial integro-differential equation (1) can thus be expressed as

$$\sum_{i=0}^{m} a_i \frac{\partial^i u(x,t)}{\partial t^i} + \sum_{i=0}^{n} b_i \frac{\partial^i u(x,t)}{\partial x^i} + cu + \sum_{i=0}^{r} d_i \int_0^t k_i(t-s) \frac{\partial^i u(x,s)}{\partial x^i} + f(x,t) = 0 \ldots \ldots(9)$$

Applying the Mahgoub transform to both sides of(9), we have

$$\sum_{i=0}^{m} a_i M\left\{\frac{\partial^i u(x,t)}{\partial t^i}\right\} + \sum_{i=0}^{n} b_i M\left\{\frac{\partial^i u(x,t)}{\partial x^i}\right\} + cM\{u\} + \sum_{i=0}^{r} d_i\, M\left\{\int_0^t k_i(t-s)\frac{\partial^i u(x,s)}{\partial x^i}\right\} + M\{f(x,t)\} = 0 \ldots \ldots \ldots..(10)$$

Using convolution theorem of Mahgoub transform and equations (5) and (8) in equation(10), we have

$$\sum_{i=0}^{m} a_i \left[v^i H(x,v) - v^i u(x,0) - v^{i-1}u_t(x,0) - \cdots \ldots - vu_{ttt \ldots.(i-1)times}(x,0)\right]$$

$+\sum_{i=0}^{n} b_i \frac{d^i H(x,v)}{dx^i} + cH(x,v)$

$+\sum_{i=0}^{r} d_i \frac{1}{v}\bar{k}_i(v)\frac{d^i H(x,v)}{dx^i} + \bar{f}(x,v) = 0 \ldots\ldots(11)$

where $M\{u(x,t)\} = H(x,v), M\{k_i(t)\} = \bar{k}_i(v)$ and $M\{f(x,t)\} = \bar{f}(x,v)$.

After using prescribed conditions, equation (11) represents an ordinary differential equation with dependent variable $H(x,v)$. After solving this ordinary differential equation and taking inverse Mahgoub transform of $H(x,v)$, we have the required solution $u(x,t)$ of equation (1).

Applications

In this section, an application is given in order to demonstrate the effectiveness of Mahgoub transform for solving linear partial integro-differential equation.

Consider the linear partial integro-differential equation [11-13]

$u_{tt} = u_x + 2\int_0^t (t-s)\, u(x,s)ds - 2e^x \ldots(12)$

with initial conditions

$u(x,0) = e^x, u_t(x,0) = 0 \ldots\ldots(13)$

and boundary condition

$u(0,t) = cost \ldots\ldots\ldots\ldots\ldots\ldots.(14)$

Applying Mahgoub transform to both sides of equation (12), we have

$$M\{u_{tt}\} = M\{u_x\} + 2M\left\{\int_0^t (t-s)\, u(x,s)ds\right\} - 2e^x M\{1\} \ldots\ldots\ldots\ldots(15)$$

Using convolution theorem of Mahgoub transform and equations (4) and (6) in equation (15), we have

$v^2 H(x,v) - v^2 u(x,0) - v u_t(x,0)$

$= \frac{dH(x,v)}{dx} + \frac{2}{v^2} H(x,v) - 2e^x \ldots\ldots\ldots(16)$

Now using equation(13) in equation (16), we have

$\frac{dH(x,v)}{dx} + \left(\frac{2}{v^2} - v^2\right) H(x,v) = (2-v^2)e^x \ldots\ldots.(17)$

which is an ordinary linear differential equation.

The general solution of equation (17) is give by

$$H(x,v) = e^x\left(\frac{v^2}{1+v^2}\right) + ce^{-\left(\frac{2}{v^2}-v^2\right)x} \ldots.(18)$$

Now, using equation (14), we have

$M\{u(0,t)\} = H(0,v) = M\{cost\} = \frac{v^2}{1+v^2} \ldots(19)$

Using equation (19) and equation(18), we have

$c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$

Substituting the value of c from equation (20) into equation (18), we have

$H(x,v) = e^x\left(\frac{v^2}{1+v^2}\right) \ldots\ldots\ldots\ldots.(21)$

Operating inverse Mahgoub transform on both sides of equation (21), we have

$$u(x,t) = M^{-1}\{H(x,v)\} = e^x M^{-1}\left\{\frac{v^2}{1+v^2}\right\} = e^x cost \ldots\ldots\ldots\ldots..(22)$$

which is the required exact solution of equation (12) with equations (13) and (*14*).

Conclusion

In this paper, we have successfully developed the Mahgoub transform for solving linear partial integro-differential equation. The given application shows that the exact solution have been obtained using very less computational work and spending a very little time. The proposed scheme can be applied for other linear partial integro-differential equations.

References

1. *Appell, J.M., Kalitvin, A.S. and Zabrejko, P.P. (2000) Partial integral operators and integro differential equations, M. Dekkar, New York.*
2. *Bahuguna, D. and Dabas, J. (2008) Existence and uniqueness of a solution to a PIDE by the method of lines, Electronic Journal of Qualatative Theory of Differential Equations, 4, 1-12.*
3. *Yanik, E.G. and Fairweather, G. (1988) Finite element methods for parabolic and hyperbolic partial integro-differential equations, Nonlinear Analysis: Theory, Method and Applications, 12, 785-809.*
4. *Dehghan, M. (2006) Solution of a partial integro-differential equation arising from viscoelasticity, International Journal of Computational Mathematics, 83, 123-129.*
5. *Sachs, E.W. and Strauss, A.K. (2008) Efficient solution of a partial integro-differential equation in finance, Applied Numerical Mathematics, 58, 1687-1703.*
6. *Mahgoub, M.A.M. (2016) The new integral transform "Mahgoub Transform", Advances in theoretical and applied mathematics, 11(4), 391-398.*
7. *Mahgoub,M.A.M. and Alshikh,A.A. (2017) An application of new transform "Mahgoub Transform" to partial differential equations, Mathematical theory and Modeling, 7(1), 7-9.*
8. *Fadhil, R.A. (2017) Convolution for Kamal and Mahgoub transforms, Bulletin of mathematics and statistics research, 5(4), 11-16.*
9. *Taha, N.E.H., Nuruddeen, R.I., Abdelilah, K. and Hassan, S. (2017) Dualities between "Kamal & Mahgoub Integral Transforms" and "Some Famous Integral Transforms", British Journal of Applied Science & Technology, 20(3), 1-8.*
10. *Zadeh, K.S. (2011) An integro-partial differential equation for modeling biofluids flow in fractured biomaterials, Journal of Theoretical Biology, 273, 72-79.*
11. *Thorwe, J. and Bhalekar, S. (2012) Solving partial integro-differential equations using Laplace transform method, American Journal of Computational and Applied Mathematics, 2(3), 101-104.*
12. *Mohand M. Abdelrahim Mahgoub and Tarig M. Elzaki (2015) Solution of partial integro-differential equations by Elzaki transform method, Applied Mathematical Sciences, 9(6), 295-303.*
13. *Aboodh, K.S., Farah, R.A., Almardy, I.A. and Almostafa, F.A. (2017) Solution of partial integro-differential equations by using Aboodh and double Aboodh transform methods, Global Journal of Pure and Applied Mathematics, 13(8), 4347-4360.*
14. *Lokenath Debnath and Bhatta, D.(2006) Integral transforms and their applications, Second edition, Chapman & Hall/CRC.*
15. *Mohand M. Abdelrahim Mahgoub (2015) Solution of partial integro-differential equations by double*

Elzaki transform method, Mathematical Theory and Modeling, 5, 61-66.

16. *Aggarwal, S., Chauhan, R. and Sharma, N. (2018) A new application of Mahgoub transform for solving linear Volterra integral equations, Asian Resonance, 7(2), 46-48.*
17. *Aggarwal, S., Sharma, N. and Chauhan, R. (2018) Solution of linear Volterra integro-differential equations of second kind using Mahgoub transform, International Journal of Latest Technology in Engineering, Management & Applied Science, 7(5), 173-176.*