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Solution of Linear Partial Integro-Differential Equations Using Mahgoub Transform

Abstract

In this paper, we used Mahgoub transform for solving linear partial integro-differential equations. The technique is described and illustrated with application. This technique gives the exact results using very less computational work.

Keywords: Linear Partial Integro-Differential Equation, Mahgoub Transform, Convolution Theorem, Inverse Mahgoub Transform.



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Introduction

Mathematical modeling of real life problems usually results in functional equations e.g. differential equations, partial differential equations, integral equations, integro-differential equations, stochastic equations, delay differential equations, partial integro-differential equations and others. In particular partial integro-differential equations arise in many scientific and engineering applications such as mathematical physics, visco-elasticity, finance, heat transfer, diffusion process, nuclear reactor dynamics, in general neutron diffusion, nano-hydrodynamics and fluid dynamics.

The general linear partial integro-differential equation is given by

$$\sum_{i=0}^m a_i \frac{\partial^i u(x, t)}{\partial t^i} + \sum_{i=0}^n b_i \frac{\partial^i u(x, t)}{\partial x^i} + cu + \sum_{i=0}^r d_i \int_0^t k_i(t, s) \frac{\partial^i u(x, s)}{\partial x^i} + f(x, t) = 0 \dots \dots (1)$$

(with prescribed conditions), where the kernels $k_i(t, s)$ and $f(x, t)$ are known functions and a_i, b_i, c and d_i are constants or functions of x . The Mahgoub transform of the function $F(t)$ is defined as [6]:

$$M\{F(t)\} = v \int_0^\infty F(t)e^{-vt} dt = H(v), t \geq 0, k_1 \leq v \leq k_2$$

where M is Mahgoub transform operator.

The Mahgoub transform of the function $F(t)$ for $t \geq 0$ exist if $F(t)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Mahgoub transform of the function $F(t)$.

Review of Literature

Appell et al. [1] discussed the partial integral operators and integro differential equations. Bahuguna and Dabas [2] gave existence and uniqueness of a solution to a PIDE by the method of lines. Yanik and Fairweather [3] used finite element methods for parabolic and hyperbolic partial integro-differential equations. Dehghan [4] discussed the solution of a partial integro-differential equation arising from viscoelasticity. Efficient solution of a partial integro-differential equation in finance was given by Sachs and Strauss [5]. Mahgoub [6] gave the new integral transform "Mahgoub Transform".

Mahgoub and Alshikh [7] applied Mahgoub transform for solving partial differential equations. Fadhil [8] gave the convolution for Kamal and Mahgoub transforms. Taha et. al. [9] gave the dualities between Kamal & Mahgoub integral transforms and some famous integral transforms. For modeling biofluids flow in fractured biomaterials, Zadeh [10] gave an integro-partial differential equation. Thorwe and Bhalekar [11] used Laplace transform method for solving partial integro-differential equations. Mohand and Tarig [12] applied Elzaki transform method for solving partial integro-differential equations. Aboodh et al. [13] gave the solution of partial integro-differential equations by using Aboodh and double Aboodh transform



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methods. Mohand [15] used double Elzaki transform method for solving partial integro-differential equations. Aggarwal et al. [16] discussed a new application of Mahgoub transform for solving linear Volterra integral equations. Aggarwal et al. [17] solved the linear Volterra integro-differential equations of second kind using Mahgoub transform.

The object of the present study is to determine exact solutions for linear partial integro-differential equations using Mahgoub transform without large computational work.

Linearity Property of Mahgoub Transforms

If $M\{F(t)\} = H(v)$ and $M\{G(t)\} = I(v)$ then $M\{aF(t) + bG(t)\} = aM\{F(t)\} + bM\{G(t)\} = aH(v) + bI(v)$, where a, b are arbitrary constants.

Mahgoub Transform of Some Elementary Functions [6, 8]

S.N.	$F(t)$	$M\{F(t)\} = H(v)$
1.	1	1
2.	t	$\frac{1}{v}$
3.	t^2	$\frac{2!}{v^2}$
4.	$t^n, n \in N$	$\frac{n!}{v^n}$
5.	e^{at}	$\frac{v}{v-a}$
6.	$\sin at$	$\frac{av}{v^2+a^2}$
7.	$\cos at$	$\frac{v^2}{v^2+a^2}$
8.	$\sinh at$	$\frac{av}{v^2-a^2}$
9.	$\cosh at$	$\frac{v^2}{v^2-a^2}$

Mahgoub transform of some partial derivatives of the function $u(x, t)$ [7]

If $M\{u(x, t)\} = H(x, v)$ then

(a) $M\left\{\frac{\partial u(x,t)}{\partial t}\right\} = vH(x, v) - vu(x, 0) \dots (3)$

(b) $M\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\} = v^2H(x, v) - v^2u(x, 0) - vu_t(x, 0) \dots (4)$

(c) $M\left\{\frac{\partial^n u(x,t)}{\partial t^n}\right\} = v^nH(x, v) - v^n u(x, 0) - v^{n-1}u_t(x, 0) - \dots - vu_{ttt \dots (n-1)times}(x, 0) \dots (5)$

(d) $M\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \frac{dH(x,v)}{dx} \dots (6)$

(e) $M\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} = \frac{d^2H(x,v)}{dx^2} \dots (7)$

(f) $M\left\{\frac{\partial^n u(x,t)}{\partial x^n}\right\} = \frac{d^n H(x,v)}{dx^n} \dots (8)$

Convolution of two functions [8]

Convolution of two functions $F(t)$ and $G(t)$ is denoted by $F(t) * G(t)$ and it is defined by

$$F(t) * G(t) = F * G = \int_0^t F(x)G(t-x)dx = \int_0^t F(t-x)G(x)dx$$

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Convolution theorem for Mahgoub transforms [8]

If $M\{F(t)\} = H(v)$ and $K\{G(t)\} = I(v)$ then

$$M\{F(t) * G(t)\} = \frac{1}{v} M\{F(t)\}M\{G(t)\} = \frac{1}{v} H(v)I(v)$$

Inverse Mahgoub Transform

If $M\{F(t)\} = H(v)$ then $F(t)$ is called the inverse Mahgoub transform of $H(v)$ and mathematically it is defined as

$$F(t) = M^{-1}\{H(v)\}$$

where M^{-1} is the inverse Mahgoub transform operator.

Inverse Mahgoub transform of some elementary functions

S.N.	$H(v)$	$F(t) = M^{-1}\{H(v)\}$
1.	1	1
2.	$\frac{1}{v}$	t
3.	$\frac{1}{v^2}$	$\frac{t^2}{2!}$
4.	$\frac{1}{v^n}, n \in N$	$\frac{t^n}{n!}$
5.	$\frac{v}{v-a}$	e^{at}
6.	$\frac{v}{v^2+a^2}$	$\frac{\sin at}{a}$
7.	$\frac{v^2}{v^2+a^2}$	$\cos at$
8.	$\frac{v}{v^2-a^2}$	$\frac{\sinh at}{a}$
9.	$\frac{v^2}{v^2-a^2}$	$\cosh at$

Mahgoub transform for linear partial integro-differential equations

In this section, we present Mahgoub transform for solving linear partial integro-differential equations given by (1). In this work, we will assume that the kernels $k_i(t, s)$ of (1) are difference kernel that can be expressed by difference $(t - s)$. The linear partial integro-differential equation (1) can thus be expressed as

$$\sum_{i=0}^m a_i \frac{\partial^i u(x, t)}{\partial t^i} + \sum_{i=0}^n b_i \frac{\partial^i u(x, t)}{\partial x^i} + cu + \sum_{i=0}^r d_i \int_0^t k_i(t-s) \frac{\partial^i u(x, s)}{\partial x^i} + f(x, t) = 0 \dots (9)$$

Applying the Mahgoub transform to both sides of (9), we have

$$\sum_{i=0}^m a_i M\left\{\frac{\partial^i u(x,t)}{\partial t^i}\right\} + \sum_{i=0}^n b_i M\left\{\frac{\partial^i u(x,t)}{\partial x^i}\right\} + cM\{u\} + \sum_{i=0}^r d_i M\left\{\int_0^t k_i(t-s) \frac{\partial^i u(x,s)}{\partial x^i}\right\} + M\{f(x, t)\} = 0 \dots (10)$$

Using convolution theorem of Mahgoub transform and equations (5) and (8) in equation (10), we have

$$\sum_{i=0}^m a_i [v^i H(x, v) - v^i u(x, 0) - v^{i-1} u_t(x, 0) - \dots - vu_{ttt \dots (i-1)times}(x, 0)]$$

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$$+ \sum_{i=0}^n b_i \frac{d^i H(x,v)}{dx^i} + cH(x,v) + \sum_{i=0}^r d_i \frac{1}{v} \bar{k}_i(v) \frac{d^i H(x,v)}{dx^i} + \bar{f}(x,v) = 0 \dots (11)$$

where $M\{u(x,t)\} = H(x,v)$, $M\{k_i(t)\} = \bar{k}_i(v)$ and $M\{f(x,t)\} = \bar{f}(x,v)$.

After using prescribed conditions, equation (11) represents an ordinary differential equation with dependent variable $H(x,v)$. After solving this ordinary differential equation and taking inverse Mahgoub transform of $H(x,v)$, we have the required solution $u(x,t)$ of equation (1).

Applications

In this section, an application is given in order to demonstrate the effectiveness of Mahgoub transform for solving linear partial integro-differential equation.

Consider the linear partial integro-differential equation [11-13]

$$u_{tt} = u_x + 2 \int_0^t (t-s)u(x,s)ds - 2e^x \dots (12)$$

with initial conditions

$$u(x,0) = e^x, u_t(x,0) = 0 \dots (13)$$

and boundary condition

$$u(0,t) = cost \dots (14)$$

Applying Mahgoub transform to both sides of equation (12), we have

$$M\{u_{tt}\} = M\{u_x\} + 2M\left\{\int_0^t (t-s)u(x,s)ds\right\} - 2e^x M\{1\} \dots (15)$$

Using convolution theorem of Mahgoub transform and equations (4) and (6) in equation (15), we have

$$v^2 H(x,v) - v^2 u(x,0) - v u_t(x,0) = \frac{dH(x,v)}{dx} + \frac{2}{v^2} H(x,v) - 2e^x \dots (16)$$

Now using equation(13) in equation (16), we have

$$\frac{dH(x,v)}{dx} + \left(\frac{2}{v^2} - v^2\right)H(x,v) = (2 - v^2)e^x \dots (17)$$

which is an ordinary linear differential equation.

The general solution of equation (17) is give by

$$H(x,v) = e^x \left(\frac{v^2}{1+v^2}\right) + ce^{-\left(\frac{2}{v^2}-v^2\right)x} \dots (18)$$

Now, using equation (14), we have

$$M\{u(0,t)\} = H(0,v) = M\{cost\} = \frac{v^2}{1+v^2} \dots (19)$$

Using equation (19) and equation(18), we have

$$c = 0 \dots (20)$$

Substituting the value of c from equation (20) into equation (18), we have

$$H(x,v) = e^x \left(\frac{v^2}{1+v^2}\right) \dots (21)$$

Operating inverse Mahgoub transform on both sides of equation (21), we have

$$u(x,t) = M^{-1}\{H(x,v)\} = e^x M^{-1}\left\{\frac{v^2}{1+v^2}\right\} = e^x cost \dots (22)$$

which is the required exact solution of equation (12) with equations (13) and (14).

Conclusion

In this paper, we have successfully developed the Mahgoub transform for solving linear partial integro-differential equation. The given application shows that the exact solution have been obtained using very less computational work and

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spending a very little time. The proposed scheme can be applied for other linear partial integro-differential equations.

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